# Review of Vectors

• **Vector Notation**: a vector quantity is one that has both *magnitude* and *direction*. Another (equivalent) way of putting it is that a vector quantity has several components in *orthogonal* (perpendicular) directions. The idea of a vector is very abstract and general; one can define useful vector spaces of many sorts, some with an infinite number of orthogonal basis vectors, but the most familiar types are simple 3-dimensional quantities like position, speed, momentum and so on. The conventional notation for a vector is $\vec{\boldsymbol{A}}$, sometimes written $\vec{\mathbf{A}}$ or $\vec{A}$ or $\boldsymbol{A}$ but most clearly recognizable when in boldface with a little arrow over the top. On the blackboard a vector may be written with a tilde *underneath*, which is hard to generate in LaTeX.

• **Unit Vectors**: In Cartesian coordinates $(x, y, z)$ a vector $\vec{\boldsymbol{A}}$ can be expressed in terms of its three scalar components $A_x, A_y, A_z$ and the corresponding unit vectors $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}$ (sometimes written as $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$ or occasionally as $\hat{\boldsymbol{x}}_1, \hat{\boldsymbol{x}}_2, \hat{\boldsymbol{x}}_3$) thus:

$$\vec{\boldsymbol{A}} = \hat{\boldsymbol{\imath}} A_x + \hat{\boldsymbol{\jmath}} A_y + \hat{\boldsymbol{k}} A_z \qquad (1)$$

where the little "hat" over a symbol means (in this context) that it has unit magnitude and thus imparts *only direction* to a scalar like $A_x$. [1]

A unit vector $\hat{\boldsymbol{a}}$ can be formed from any vector $\vec{\boldsymbol{a}}$ by dividing it by its own *magnitude* $a$:

$$\hat{\boldsymbol{a}} = \frac{\vec{\boldsymbol{a}}}{a} \qquad \text{where} \quad a = |\vec{\boldsymbol{a}}| = \sqrt{a_x^2 + a_y^2 + a_z^2}\,. \qquad (2)$$

Already we have used a bunch of concepts before defining them properly, the usual chicken-egg problem with mathematics. Let's try to catch up:

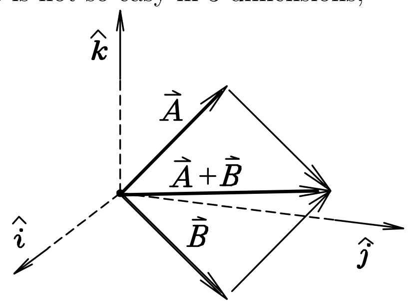
• **Multiplying or Dividing a Vector by a Scalar**: Multiplying a vector $\vec{\boldsymbol{A}}$ by a scalar $b$ has no effect on the *direction* of the result (unless $b = 0$) but only on its *magnitude* and/or the *units* in which it is measured — if $b$ is a pure number, the units stay the same; but multiplying a velocity by a mass (for instance) produces an entirely new quantity, in that case the momentum.

Dividing a vector by a scalar $c$ is the same as *multiplying* it by $1/c$.

This type of product always commutes: $\vec{\boldsymbol{A}} b = b \vec{\boldsymbol{A}}$.

[1] There are many choices of coordinates and unit vectors, such as *cylindrical* $(r, \theta, z)$ and *spherical* $(r, \theta, \phi)$ coordinates, but only in the simple Cartesian coordinates are the directions of the unit vectors permanently fixed.

• **Adding or Subtracting Vectors**: In two dimensions one can draw simple diagrams depicting "tip-to-tail" or "parallelogram law" vector addition (or subtraction); this is not so easy in 3 dimensions,




so we fall back on the algebraic method of *adding components*. Given $\vec{\boldsymbol{A}}$ from Eq. (1) and

$$\vec{\boldsymbol{B}} = \hat{\boldsymbol{\imath}} B_x + \hat{\boldsymbol{\jmath}} B_y + \hat{\boldsymbol{k}} B_z \qquad (3)$$

we write

$$\vec{\boldsymbol{A}} + \vec{\boldsymbol{B}} = \hat{\boldsymbol{\imath}}(A_x + B_x) + \hat{\boldsymbol{\jmath}}(A_y + B_y) + \hat{\boldsymbol{k}}(A_z + B_z)\,. \qquad (4)$$

Subtracting $\vec{\boldsymbol{B}}$ from $\vec{\boldsymbol{A}}$ is the same thing as adding $-\vec{\boldsymbol{B}}$.

• **Multiplying Two Vectors . . .**

**. . . to get a Scalar**: we just add together the products of the components,

$$\begin{aligned} \vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} &= A_x B_x \\ &+ A_y B_y \\ &+ A_z B_z\,, \end{aligned} \qquad (5)$$

also known as the "dot product", which commutes: $\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} = \vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{A}}$.

**. . . to get a Pseudovector**:

$$\begin{aligned} \vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}} &= \hat{\boldsymbol{\imath}}(A_y B_z - A_z B_y) \\ &+ \hat{\boldsymbol{\jmath}}(A_z B_x - A_x B_z) \\ &+ \hat{\boldsymbol{k}}(A_x B_y - A_y B_x)\,. \end{aligned} \qquad (6)$$

This "cross product" is actually a pseudovector (or, more generally, a tensor), because (unlike the nice dot product) it has the unsettling property of *not commuting* ($\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}} = -\vec{\boldsymbol{B}} \times \vec{\boldsymbol{A}}$) but we often treat it like just another vector.