The University of British Columbia

Physics 401 Assignment # 4: POTENTIALS, GAUGES and RELATIVITY

SOLUTIONS:

Wed. 25 Jan. 2006 — finish by Wed. 1 Feb.

Please review Section 10.1 and Ch. 12.

1. (p. 420, Problem **10.3**) — **GIVEN** V **&** $\vec{A}$ **. . .** Find the $\vec{E}$, $\vec{B}$, ρ & $\vec{J}$ corresponding to

$$V(\vec{r},t) = 0, \qquad \vec{A}(\vec{r},t) = -\frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\,\hat{r}\,.$$

ANSWER: $\boxed{\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t} = \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}}$. $\vec{B} = \vec{\nabla}\times\vec{A} = \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi}\right]\hat{\theta} - \frac{1}{r}\left[\frac{\partial A_r}{\partial\theta}\right]\hat{\phi}$ but A_r varies only with r, not with ϕ or θ, so $\boxed{\vec{B} = 0}$. These are the $\vec{E}$ and $\vec{B}$ of a stationary point charge at the origin, so $\boxed{\vec{J} = 0}$ and $\boxed{\rho = q\,\delta(\vec{r})}$.

2. **POINT CHARGE:**

(*a*) Find the $\vec{E}$ and $\vec{B}$ fields corresponding to a stationary point charge q situated at the origin. **ANSWER**: This is a freebie.

(*b*) State the charge and current distributions of this situation. **ANSWER**: See above.

(*c*) What are the electric and magnetic potentials? **ANSWER**: We have learned to use $\boxed{V = \frac{q}{4\pi\epsilon_0}\frac{1}{r} \text{ and } \vec{A} = 0}$. Those potentials are certainly not *wrong*, but they don't look like the ones in Problem **10.3**!

(*d*) Is there any relation between this situation and that described in Problem **10.3**? **ANSWER**: Well, obviously, since they give the same fields, they describe exactly the same physics! This is a vivid illustration of gauge invariance: an infinite variety of choices for V and $\vec{A}$ will all give the same $\vec{E}$ and $\vec{B}$ as long as they differ only by adding the gradient of some scalar function to $\vec{A}$ while subtracting the time derivative of the same function from V.

3. (p. 420, Problem **10.5**) — **GAUGE TRANSFORMATION:** Use the gauge function

$$\lambda = -\frac{1}{4\pi\epsilon_0}\left(\frac{q\,t}{r}\right)$$

to transform the potentials in Problem **10.3**, and comment on the result.
ANSWER: $\vec{A}'(\vec{r},t) = \vec{A}(\vec{r},t) + \vec{\nabla}\lambda$ and $\vec{\nabla}\lambda = \frac{qt}{4\pi\epsilon_0}\frac{\hat{r}}{r^2} = -\vec{A}(\vec{r},t)$ so $\boxed{\vec{A}'(\vec{r},t) = 0}$ as in Problem 2. Meanwhile $V'(\vec{r},t) = V(\vec{r},t) - \partial\lambda/\partial t$ where $V(\vec{r},t) = 0$ and $\frac{\partial\lambda}{\partial t} = -\frac{q}{4\pi\epsilon_0 r}$ so $V'(\vec{r},t)$ is just the potential due to a point charge q at the origin, again as in Problem 2. So λ is just the gauge function required to transform between these two descriptions of the same thing.

4. **WHICH GAUGE?**

(*a*) In Problem **10.3** above, are the potentials in the Coulomb gauge, the Lorentz gauge, both, or neither? **ANSWER**: With $V(\vec{r},t) = 0$ we have $\partial V/\partial t = 0$ while $\vec{\nabla}\cdot\vec{A} = -\frac{qt}{4\pi\epsilon_0}\,\delta(\vec{r}) = \frac{t}{4\pi\epsilon_0}\,\rho$. This is neither zero (Coublob gauge) nor $-\partial V/\partial t$ (Lorentz gauge), so the answer is $\boxed{\text{neither}}$.

(*b*) In Problem 2 above, are the potentials in the Coulomb gauge, the Lorentz gauge, both, or neither? **ANSWER**: In this case V is constant in time, so again $\partial V/\partial t = 0$; meanwhile, if $\vec{A} = 0$ then $\vec{\nabla}\cdot\vec{A} = 0 = -\partial V/\partial t$, so the answer in this case is $\boxed{\text{both}}$. It's important to note that "being in" one gauge does *not* preclude "being in" another!

5. **NATURAL UNITS:** Since c is now a *defined* quantity that keeps appearing in confusing places in our notation for 4-vectors *etc.*, and since *nanoseconds* (ns) are perfectly handy units for *distance*, it seems silly to not just measure time and distance in the same units (seconds) and set $c = 1$. While we're at it, why not set the ubiquitous constant in quantum mechanics to unity as well ($\hbar = 1$) so that all angular momenta are unitless and (because $E = \hbar\omega$) energies are measured in s^{-1}?

(*a*) In what units would we then measure velocities, momenta, masses, forces and accelerations? **ANSWER**: **Velocities** are of course $\boxed{\text{unitless}}$ ($v \equiv \beta$). **Momenta** are part of the same 4-vector as energies, and thus differ from them only by factors of $c = 1$ (*e.g.* for massless particles $E = pc$), so momenta are also measured in $\boxed{\text{s}^{-1}}$. **Masses** are also measured in $\boxed{\text{s}^{-1}}$, since $E = mc^2$. **Forces** have the same units as the time rates of change of momenta, namely $\boxed{\text{s}^{-2}}$. **Accelerations** are the time rate of change of velocities or forces divided by masses; either way, they are measured in $\boxed{\text{s}^{-1}}$.

(*b*) Suppose we set the Coulomb force constant $k_E \equiv \frac{1}{4\pi\epsilon_0} = 1$ as well. In what units would we then measure charge, electric field, magnetic field, and potentials V and $\vec{\boldsymbol{A}}$? **ANSWER**: COULOMB'S FORCE LAW is now written $\vec{\boldsymbol{F}} = q_1 q_2 \hat{\boldsymbol{r}}/r^2$, so [charge]^2 = [force]$\times$[distance]^2 so **Charge** is $\boxed{\text{unitless}}$. **Electric field** is force per unit charge, so it now has the *same* units as force, namely $\boxed{\text{s}^{-2}}$. Alternatively, E^2 is proportional to energy per unit volume. Volume is in s^3, so E^2 is in s^{-4}. Thus again E is in s^{-2}. **Magnetic field** is in $\boxed{\text{s}^{-2}}$ the same as $\vec{\boldsymbol{E}}$, because B^2 is also proportional to energy per unit volume.[1] V is $\int \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{\ell}}$ and ℓ is in seconds, since velocities are unitless; so V is in $\boxed{\text{s}^{-1}}$. (This can also be deduced from $U = qV$.) $\vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}$ so $\vec{\boldsymbol{A}}$ has one less s in its denominator: it is measured in $\boxed{\text{s}^{-1}}$ just like V. (This should not be surprising.)

(*c*) Write out Maxwell's equations in this system of units. (*Hint:* We must have $\epsilon_0\,\mu_0 = 1$.)[2] **ANSWER**: We have set $\epsilon_0 = 1/4\pi$ so we must set $\mu_0 = 4\pi$ to ensure $c = 1$. Then COULOMB'S LAW reads $\vec{\boldsymbol{E}} = q\hat{\boldsymbol{r}}/r^2$ which, integrated over a sphere centred on the charge, gives $\oint\!\!\!\oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{a}} = (4\pi r^2) \times (q/r^2) = 4\pi q$, so GAUSS' LAW has a 4π where the $(1/\epsilon_0)$ used to be, just as expected. By the same token we expect to replace μ_0 by 4π in the AMPÈRE/MAXWELL LAW; so Maxwell's equations become just

$$\begin{aligned} \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}} &= 4\pi\rho & \quad \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}} &= -\frac{\partial \vec{\boldsymbol{B}}}{\partial t} \\ \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{B}} &= 0 & \quad \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}} &= 4\pi \vec{\boldsymbol{J}} + \frac{\partial \vec{\boldsymbol{E}}}{\partial t} \end{aligned}$$

where both sides of each equation are measured in units of s^{-3}. These are of course the "microscopic" equations; if we want to use the "macroscopic average" fields $\vec{\boldsymbol{H}} = \vec{\boldsymbol{B}}/\mu$ and $\vec{\boldsymbol{D}} = \epsilon\vec{\boldsymbol{E}}$, we must include a dimensionless permittivity $\epsilon \equiv \kappa$ (where κ is the dielectric constant) and dimensionless permeability $\mu \equiv 1 + \chi_m$ (where χ_m is the magnetic susceptibility).

That was pretty simple. But before you get too enthusiastic, you might want to make a *quantitative* calculation of *your own weight* or *the electron's charge* in these units. For the former I get something like 10^{58} s^{-2}. (I wouldn't want to have to paint the dials on bathroom scales if the world embraced these particular "natural units"!)[3] One can also set Newton's universal gravitational constant $G = 1$ to define PLANCK UNITS, but I don't understand how that works. We already have everything covered, as far as I can see.

6. 4-POTENTIAL: In Eq. (12.131) on p. 541, Griffiths states that, "As you might guess, V and $\vec{\boldsymbol{A}}$ together constitute a 4-vector: $A^\mu = (V/c, A_x, A_y, A_z)$." This is a very strong statement with profound consequences. You can't just take any 3-vector and combine it with a convenient scalar in the same units to make a true 4-vector! *Explain why* we should believe this about A^μ, and list any essential *conditions* that must be met for it to be true.
ANSWER: If, as also stated in Griffiths, $\partial_\mu \equiv \partial/\partial x^\mu$ is a covariant 4-vector, then its inner product with A^μ would be $\partial_\mu A^\mu = \frac{1}{c^2}\frac{\partial V}{\partial t} + \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}}$. But the LORENTZ GAUGE explicitly sets the RHS of this equation to zero, and zero is certainly a Lorentz invariant scalar. This inner product is therefore invariant, so A^μ must be a *bona fide* 4-vector, as long as we stay in the LORENTZ GAUGE.[4]

[1] You might worry about the constant of proportionality, but if ϵ_0 is a pure number then μ_0 must be one also, or c would not be unitless.

[2] You are allowed to consult the literature on this, or even `http://en.wikipedia.org/wiki/Planck_units`, but please explain your own reasoning!)

[3] Maybe the "googol" (10^{100}) has a practical use after all! If we take its square root (10^{50}, a "roogol"?) we're in a convenient range for lots of things. New prefixes: R and r such that one roogolhertz = 1 RHz $\equiv 10^{50}$ Hz and one reallylittlesecond = 1 rs $\equiv 10^{-50}$ s. No problem.

[4] I find this distinctly upsetting. The 4-vector-ness of A^μ depends on an essentially arbitrary choice of gauge?! Surely something as fundamental as whether a 4-quantity is a true 4-vector or not should not be subject to arbitrariness! And yet we cannot just declare one gauge to be the "right" gauge, since no measureable quantity depends upon the choice of gauge. I hope you can think of a resolution to this conundrum.