THE UNIVERSITY OF BRITISH COLUMBIA

Physics 401 Assignment # 11: RADIATION *1* SOLUTIONS:

Wed. 22 Mar. 2006 — finish by Wed. 29 Mar.

1. (p. 449, Problem **11.2**) — **Electric Dipolar Radiation:** Equation (11.14),

$$V(r,\theta,t) = -\frac{p_0\omega}{4\pi\epsilon_0 c}\left(\frac{\cos\theta}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right],$$

can be expressed in "coordinate-free" form by writing $p_0\cos\theta = \vec{\boldsymbol{p}}_0\cdot\hat{\boldsymbol{r}}$. Do so...

ANSWER: $\boxed{V = -\frac{\omega}{4\pi\epsilon_0 c}\left(\frac{\vec{\boldsymbol{p}}_0\cdot\hat{\boldsymbol{r}}}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]}$

... and similarly for Equation (11.17),

$$\vec{\boldsymbol{A}}(r,\theta,t) = -\frac{\mu_0 p_0\omega}{4\pi r}\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\boldsymbol{z}},$$

ANSWER: $\boxed{\vec{\boldsymbol{A}} = -\frac{\mu_0\vec{\boldsymbol{p}}_0\omega}{4\pi r}\sin\left[\omega\left(t-\frac{r}{c}\right)\right]}$

... Equation (11.18),

$$\vec{\boldsymbol{E}}(r,\theta,t) = -\frac{\mu_0 p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\boldsymbol{\theta}},$$

ANSWER:

$$\boxed{\vec{\boldsymbol{E}} = \frac{\mu_0\omega^2}{4\pi}\left(\frac{(\hat{\boldsymbol{r}}\times\vec{\boldsymbol{p}}_0)\times\hat{\boldsymbol{r}}}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]}$$

... Equation (11.19),

$$\vec{\boldsymbol{B}}(r,\theta,t) = -\frac{\mu_0 p_0\omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\boldsymbol{\phi}},$$

ANSWER:

$$\boxed{\vec{\boldsymbol{B}} = \frac{\mu_0\omega^2}{4\pi c}\left(\frac{\hat{\boldsymbol{r}}\times\vec{\boldsymbol{p}}_0}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]} \quad \ldots$$

and Equation (11.21),

$$\langle\vec{\boldsymbol{S}}\rangle = \left(\frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\right)\frac{\sin^2\theta}{r^2}\hat{\boldsymbol{r}}.$$

ANSWER: As always,

$$\vec{\boldsymbol{S}} \equiv \frac{1}{\mu_0}\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}} = \frac{\omega^2}{4\pi}\cdot\frac{\mu_0\omega^2}{4\pi c}\cdot$$

$$\cdot\frac{[(\hat{\boldsymbol{r}}\times\vec{\boldsymbol{p}}_0)\times\hat{\boldsymbol{r}}]\times[\hat{\boldsymbol{r}}\times\vec{\boldsymbol{p}}_0]}{r^2}\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]$$

$$\text{or}\quad \langle\vec{\boldsymbol{S}}\rangle = \frac{\mu_0\omega^4}{32\pi^2 c}\left(\frac{|\hat{\boldsymbol{r}}\times\vec{\boldsymbol{p}}_0|^2}{r^2}\right)\hat{\boldsymbol{r}}.$$

2. **Atomic Dipoles:** Explain why you can safely assume $\frac{\vec{\boldsymbol{m}}_0}{c} \ll \vec{\boldsymbol{p}}_0$ for an atom with magnetic dipole moment $\vec{\boldsymbol{m}}_0$ and electric dipole moment $\vec{\boldsymbol{p}}_0$, assuming typical values of relevant physical quantities. **ANSWER**: Using Bohr's model of the H atom, we have an electron ($m_e \simeq 0.911\times10^{-30}$ kg, $e \simeq 1.6\times10^{-19}$ C) orbiting a heavy nucleus at radius $a_0 \simeq 0.53\times10^{-10}$ m, with angular momentum $L = m_e a_0^2\omega = \hbar \simeq 1.05\times10^{-34}$ kgm^2/s. Thus $\omega = \hbar/m_e a_0^2 \simeq 4.13\times10^{16}$ s^{-1}. The amplitude of the electric dipole moment (in the plane of the orbit) is thus $p_0 = ea_0 \simeq 0.85\times10^{-29}$ Cm. The magnetic dipole moment is $m_0 = \pi a_0^2(e\omega/2\pi) \simeq 0.927\times10^{-23}$ m^{2}C/s, and $m_0/c \simeq 3.09\times10^{-30}$ Cm, which is a factor of $\boxed{274}$ smaller than p_0.

3. (p. 473-474, Problem **11.22**) — **Broadcasting KRUD:** A radio tower rises to a height h above flat horizontal ground. At the top is a magnetic dipole antenna of radius b, with its axis vertical. FM station KRUD broadcasts from this antenna at angular frequency ω, with a total radiated power P (averaged, of course, over a full cycle). Neighbors have complained about problems they attribute to excessive radiation from the tower — interference with their stereo systems, mechanical garage doors opening and closing mysteriously, and a variety of suspicious medical problems. But the city engineer who measured the radiation at the base of the tower found it to be well below the accepted standard. You have been hired by the Neighborhood Association to assess the engineer's report.

(*a*) In terms of the variables given (not all of which may be relevant, of course) find the formula for the intensity of the radiation at ground level, a distance R away from the base of the tower. You may assume that $b \ll c/\omega \ll h$. [*Note:* we are interested only in the *magnitude* of the radiation, not in its *direction* — when measurements are taken, the detector will be aimed directly at the antenna.]

ANSWER: The average energy flux from a magnetic dipole antenna, Eq. (11.39), is

$$\langle\vec{\boldsymbol{S}}\rangle = \left(\frac{\mu_0 m_0^2\omega^4}{32\pi^2 c^3}\right)\frac{\sin^2\theta}{r^2}\hat{\boldsymbol{r}}.$$

The total power radiated, Eq. (11.40), is

$$\langle P\rangle = \left(\frac{\mu_0 m_0^2\omega^4}{12\pi c^3}\right).$$

Thus $\langle\vec{\boldsymbol{S}}\rangle = \left(\frac{3\langle P\rangle \sin^2\theta}{8\pi r^2}\right)\hat{\boldsymbol{r}}\,.$

Here we have $r^2 = h^2 + R^2$ and $\sin^2\theta = R^2/r^2$, giving

$$\boxed{|\langle\vec{\boldsymbol{S}}\rangle| = \frac{3\langle P\rangle R^2}{8\pi(h^2+R^2)^2}}\,.$$

The intensity at the *base* of the tower ($R = 0$) is zero, *Doh!*

(*b*) How far from the base of the tower *should* the engineer have made the measurement? What is the formula for the intensity at this location? **ANSWER**: At the point of highest intensity, of course. As for any extremum, this requires

$$\frac{\partial\,|\langle\vec{\boldsymbol{S}}\rangle|}{\partial R} = 0 = \frac{3\langle P\rangle}{8\pi}\frac{\partial}{\partial R}\left(\frac{R^2}{(h^2+R^2)^2}\right)$$

$$\text{or}\quad \frac{2R}{(h^2+R^2)^2} - \frac{4R^3}{(h^2+R^2)^3} = 0$$

$$\text{or}\quad \frac{2R^2}{h^2+R^2} = 1 \quad\text{or}\quad \boxed{R = h}\,.$$

(*c*) KRUD's actual power output is 35 kilowatts, its frequency is 90 MHz, the antenna's radius is 6 cm, and the height of the tower is 200 m. The city's radio-emission limit is 200 microwatts/cm^2. Is KRUD in compliance? **ANSWER**: We don't need to know ω or b. All we need is $\langle P\rangle = 3.5\times10^4$ W and $h = 200$ m. At the point of highest intensity (at ground level), $R = h$, we have

$$|\langle\vec{\boldsymbol{S}}\rangle| = \frac{3\langle P\rangle h^2}{8\pi(h^2+h^2)^2} = \frac{3\times3.5\times10^4}{32\pi(200)^2}$$

$$\text{or}\quad \boxed{|\langle\vec{\boldsymbol{S}}\rangle|_{\max} = 0.02611\ \text{W/m}^2}$$

$= 2.611\ \mu$W/cm^2, well within compliance.[1]

4. (p. 474, Problem **11.23**) — **Earth as a Pulsar:** The magnetic north pole of the Earth does not coincide with the geographic North Pole — in fact, it's off by about 7° at present.[2] Relative to the fixed axis of rotation, therefore, the magnetic dipole moment vector of the Earth is changing with time, so the Earth must be giving off magnetic dipole radiation.

(*a*) Find the formula for the total power radiated, in terms of the following parameters: Ψ (the angle between the geographic and magnetic north poles), M (the magnitude of the Earth's magnetic dipole moment), and ω (the angular velocity of rotation of the Earth). [*Hint:* refer to Prob. 11.4 or Prob. 11.12.]

ANSWER: Problem 11.12 gives the total power radiated by a magnetic dipole generated by a time-varying current in a circular loop: $P_{\rm loop} = \mu_0\ddot{m}^2/6\pi c^3$. Problem 11.4 describes an *electric* dipole **rotating** about the $\hat{\boldsymbol{z}}$ axis as a superposition of two **oscillating** dipoles in the $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ directions, $\pi/2$ out of phase: $\vec{\boldsymbol{p}}(t) = p_0[\cos(\omega t)\hat{\boldsymbol{x}} + \sin(\omega t)\hat{\boldsymbol{y}}]$. You are then invited to find the intensity as a function of the polar angle θ and calculate the total power radiated, explaining why the power seems to satisfy the superposition principle even though it is quadratic in the fields.

A more conventional way to represent *precession of a dipole* is to make the $\hat{\boldsymbol{x}}$ component real and the $\hat{\boldsymbol{y}}$ component imaginary: $\tilde{m}(t) = m_0 e^{i\omega t}$ which amounts to the same thing as above. In the Earth's case it is only the transverse component $M_\perp = M\sin\Psi\, e^{i\omega t}$ that precesses; the axial component $M_\| = M\cos\Psi$ just adds a constant magnetic dipole field. Thus $\ddot{m} = -\omega^2 M\sin\Psi\, e^{i\omega t}$ and we expect[3]

$$\boxed{P = \frac{\mu_0\,\omega^4\,M^2\,\sin^2\Psi}{6\pi c^3}}\,.$$

[1] Of course, if you report this to the Neighborhood Association, they will fire you, accuse you of being a pawn of Big Broadcasting, and hire someone else to give them the answer they want.

[2] The disagreement between the current value and that in Griffiths is due to the fact that magnetic north pole (which is actually a **south** magnetic pole, of course) has been drifting approximately northwest at about 40 km per year for the last few years (a blistering pace on a geological time scale); it has always wandered around like this, and has reversed direction more than once! Sailors (and students in Power Squadron courses) must learn how to correct their compass readings for this gradual drift.

[3] When using complex notation $m_0e^{i\omega t}$ to describe precession, we must understand $\ddot{m}^2$ to mean $|\ddot{m}|^2$ — *i.e.* the $e^{i\omega t}$ term is multiplied by its complex conjugate, $e^{-i\omega t}$ to give unity for the time dependence. This expresses the conclusion of Problem 11.12: the dipole moment never gets larger or smaller, it just changes direction, and the radiation pattern follows it. If you stood on top of it and rotated with it (as we certainly do on the Earth) you would see a fixed intensity profile and the fact that it radiates at all would be confusing if you failed to notice that you were in a rotating reference frame.

(*b*) Using the fact that the Earth's magnetic field is about half a gauss at the Equator, estimate the magnetic dipole moment M of the Earth. **ANSWER**: From Eq. (5.87) on p. 246 we have the field of a static magnetic dipole:

$$\vec{\boldsymbol{B}}_{\rm dip} = \frac{\mu_0}{4\pi}\left[\frac{3(\vec{\boldsymbol{m}}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \vec{\boldsymbol{m}}}{r^3}\right]$$

which reduces to $\vec{\boldsymbol{B}}_{\rm dip} = -\mu_0\vec{\boldsymbol{m}}/4\pi r^3$ for $\hat{\boldsymbol{r}} \perp \vec{\boldsymbol{m}}$ (*i.e.* at the Equator). Thus $M \approx 4\pi \times 0.5 \times 10^{-4} \times (6.4 \times 10^6)^3/\mu_0$ or $\boxed{M \approx 1.3 \times 10^{23} \text{ Am}^2}$. (We neglect the 7° tilt and the fact that the Earth's dipole is far from pointlike on the scale of R_E.)

(*c*) Find the power radiated. [Your answer should be several times 10^{-5} W.]
ANSWER: The Earth's $\omega = 2\pi/(24 \times 60 \times 60) \simeq 0.727 \times 10^{-4}$ s^{-1}. Plugging this, $M \approx 1.3 \times 10^{23}$ Am^2, and $\sin\Psi \approx \Psi = 0.122$ into

$$P = \frac{\mu_0\,\omega^4\,M^2\,\sin^2\Psi}{6\pi c^3}$$

gives $\boxed{P_{\rm Earth} = 1.73 \times 10^{-5} \text{ W}}$.

(*d*) Pulsars are thought to be rotating neutron stars, with a typical radius of about $R \sim 10$ km, a typical surface magnetic field of $B(R) \sim 10^8$ T and a variety of rotational periods T; let's use $T \sim 10^{-3}$ s. What sort of radiated power would you expect from such a star? [See J.P. Ostriker and J.E. Gunn, *Astrophys. J.* **157**, 1395 (1969). *Answer:* 2×10^{36} W.]
ANSWER: Again we use $B_{\rm dip}^{\rm Equator} = \mu_0\,m/4\pi r^3$ to get $m \approx 4\pi \times 10^8 \times (10^4)^3/\mu_0 = 10^{27}$ Am^2. This is only some 8000 times bigger than the Earth's magnetic moment, but the frequency $\omega \approx 2\pi/10^{-3} = 6300$ s^{-1} is a lot bigger, and ω^4 is ... well ... **huge**. Thus (assuming the star's magnetic moment is perpendicular to its axis of rotation, which gives the biggest result) we get $\boxed{P_{\rm star} \approx 3.86 \times 10^{36} \text{ W}}$! This is about a factor of two larger than the value predicted by Griffiths. No doubt this is because of the probability distribution of angles Ψ between the star's magnetic moment and its axis of rotation. If we assume all values of Ψ between 0 and π are equally likely, then we should multiply our result by $\langle\sin^2\Psi\rangle = 1/2$, giving $\boxed{\langle P_{\rm star}\rangle \approx 1.93 \times 10^{36} \text{ W}}$. But this seems a little silly in two respects: first, we are just making an estimate for a "typical" neutron star. A 20% change of ω would have the same effect. Second, it seems improbable that the formation of neutron stars from supernovae of spinning suns would indiscriminately orient the star's magnetic moment relative to its axis or rotation; naively one might expect $\Psi \sim 0$ to be more likely, which would bias our estimate toward much smaller values of $\langle P_{\rm star}\rangle$. A more realistic estimate would require a deeper knowledge of astrophysics than I (for one) possess.[4]

[4]The magnetism of neutron stars is a very interesting topic. If you assume (unrealistically) a pointlike central magnetic dipole, the field at 1 km from the centre of this "typical" star is more like 10^{11} T, and others may have much higher fields. Thus some may reach fields $\sim 10^{12}$ T at which there is speculation that the Cabbibo angle (one of the key quantities in the so-called "Standard Model" of elementary particle physics) might vanish! [See A. Salam and J. Strathdee, *Nature* **252**, 569 (1974).]

On Earth we don't usually encounter such large magnetic fields, unless you count the *effective* field $B_{\rm eff} \sim 5 \times 10^{10}$ T of a magnetic nucleus like ^{93}Nb at a negative muon bound to it in an orbit that is mostly inside the nucleus; this might explain the anomalously large nuclear capture rate of the μ^- on that nucleus. [See P.J.S. Watson, *Phys. Lett.* **58B**, 431 (1975).]