UBC Physics 401 Lecture:

DIPOLE RADIATION

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http://musr.physics.ubc.ca/p401/pdf/DipoleRadPlus.pdf

OUTLINE

- **Summary of Electric Dipole Radiation**

- **Magnetic Dipole Radiation**
  * Far Field approximation
  * Slow approximation
  * Distant approximation
  * Summary of Magnetic Dipole Radiation

- **Electric *vs.* Magnetic Dipole Antennas**
  * Size Matters!

Summary of Electric Dipole Radiation

$$\vec{\boldsymbol{E}}(\vec{r},t) = -\frac{\mu_0\, p_0\, \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\ \hat{\boldsymbol{\theta}} \tag{1}$$

$$\vec{\boldsymbol{B}}(\vec{r},t) = -\frac{\mu_0\, p_0\, \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\ \hat{\boldsymbol{\phi}} \tag{2}$$

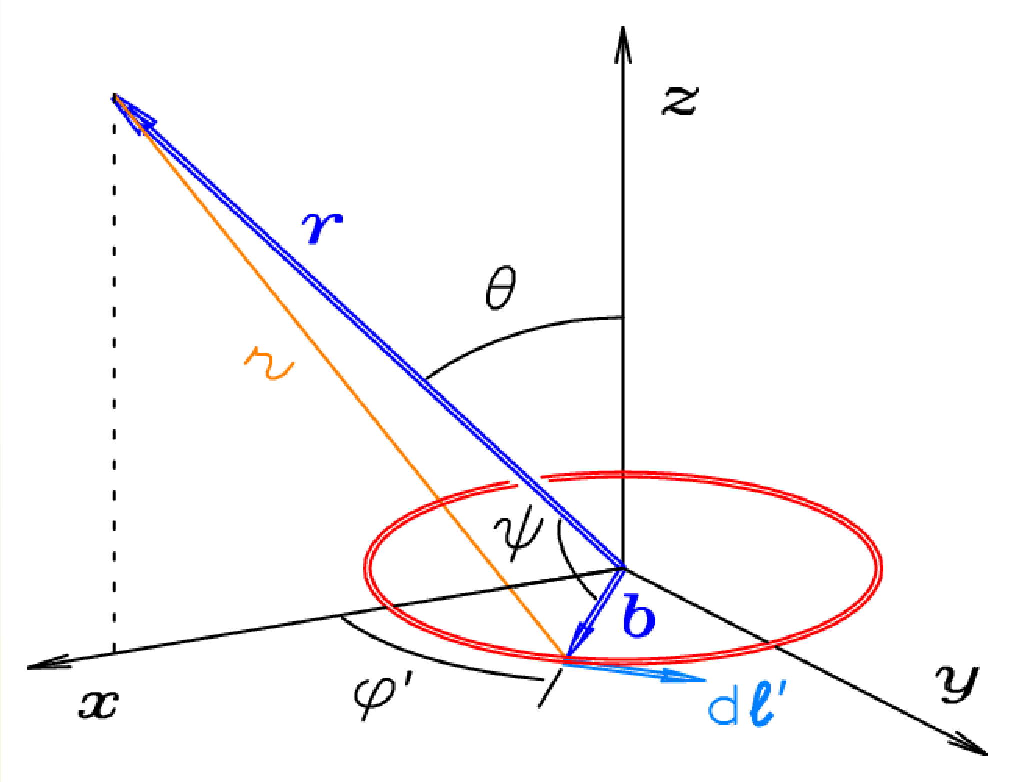
$$\langle\vec{\boldsymbol{S}}\rangle = \frac{\mu_0\, p_0^2\, \omega^4}{32\pi^2 c}\left(\frac{\sin^2\theta}{r^2}\right)\ \hat{\boldsymbol{r}} \tag{3}$$

$$\langle P\rangle = \frac{\mu_0\, p_0^2\, \omega^4}{12\pi c} = \frac{\mu_0\, q^2\, a^2}{6\pi c} \quad \text{since} \quad \vec{\boldsymbol{p}} = q\vec{\boldsymbol{d}} \Longrightarrow \ddot{\boldsymbol{p}} = q\vec{\boldsymbol{a}} \Longrightarrow p_0\omega^2 = qa \tag{4}$$

(This is the famous **Larmor formula**.)

OUTLINE

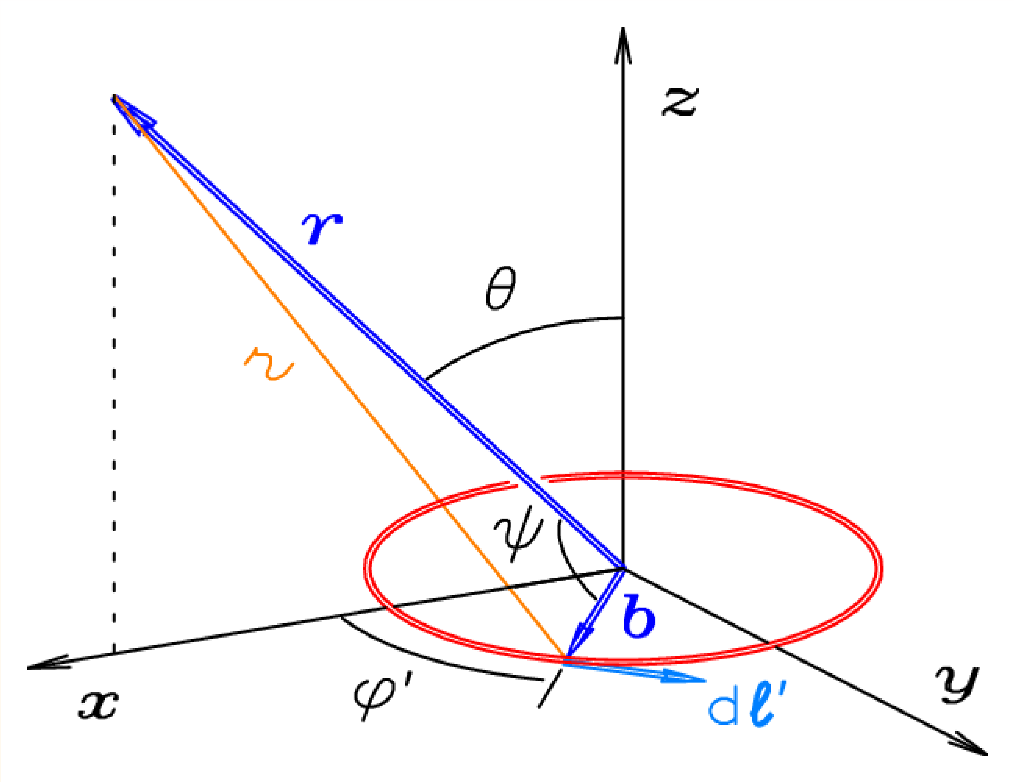
Magnetic Dipole Radiation



With the coordinate system defined at left, the magnetic dipole moment is $\vec{m} = m\hat{z}$ where $m = IA$ and A is the area of the loop. For a circular loop of radius b carrying a current I, then, $m = \pi b^2 I$. If I is oscillating at frequency ω with amplitude I_0, $m(t) = m_0 \cos\omega t$ where $m_0 = \pi b^2 I_0$.

We start with the **retarded potential**,

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \oint \frac{I(t_r) d\vec{\ell}\,'}{\mathcal{R}} \tag{5}$$



The retarded potential at $\vec{\boldsymbol{r}}$ is

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}},t) = \frac{\mu_0}{4\pi}\oint \frac{I(t_r)d\vec{\boldsymbol{\ell}}\,'}{\mathcal{R}}\,.$$

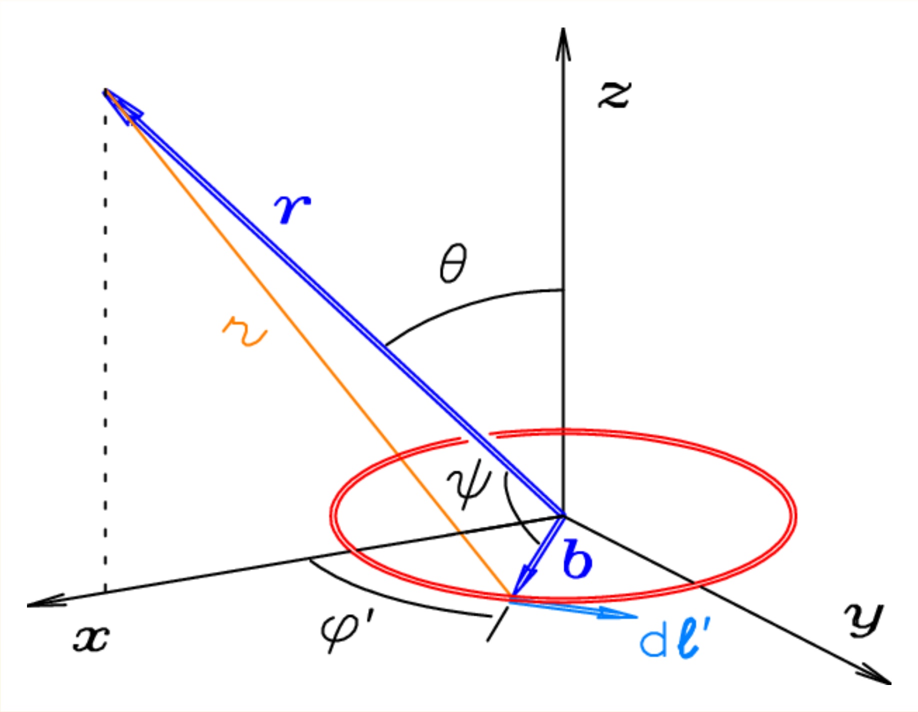
For each element $d\vec{\boldsymbol{\ell}}\,'$ on the $+y$ side there is an equal and opposite $d\vec{\boldsymbol{\ell}}\,'$ on the $-y$ side at the same distance $\mathcal{R}$, so the $\hat{\boldsymbol{x}}$-components *cancel*, leaving only the $\hat{\boldsymbol{y}}$-component of each $d\vec{\boldsymbol{\ell}}\,'$:

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}},t) = \frac{\mu_0 I_0 b}{4\pi}\,\hat{\boldsymbol{y}}\int_0^{2\pi} \frac{\cos\left[\omega\left(t-\frac{\mathcal{R}}{c}\right)\right]}{\mathcal{R}}\,\cos\phi'\,d\phi' \qquad (6)$$

We can substitute $\hat{\boldsymbol{\phi}}$ for $\hat{\boldsymbol{y}}$ to free this description from the particular choice of x and y axes shown here. The next step is to *approximate* $\mathcal{R}$ as before . . .

OUTLINE

Far Field approximation



By the *Law of Cosines*, we have

$$\mathcal{R}^2 = r^2 + b^2 - 2rb\cos\Psi \ , \quad \text{so} \tag{7}$$

$$\mathcal{R} = r\sqrt{1 + \frac{b^2}{r^2} - \frac{2b}{r}\sin\theta\,\cos\phi'} \tag{8}$$

Approximation 1: $\boxed{d^2 \ll r^2}$
("**Far Field**" approximation):

$$\text{Then} \qquad \mathcal{R} \approx r\left[1 - \frac{2b}{r}\sin\theta\,\cos\phi')\right]^{1/2} \approx r\left[1 - \frac{b}{r}\sin\theta\,\cos\phi'\right] \tag{9}$$

$$\text{and} \qquad \frac{1}{\mathcal{R}} \approx \frac{1}{r}\left[1 - \frac{2b}{r}\sin\theta\,\cos\phi')\right]^{-1/2} \approx \frac{1}{r}\left[1 + \frac{b}{r}\sin\theta\,\cos\phi'\right] \tag{10}$$

OUTLINE

Slow approximation

Approximation 2: $\boxed{b \ll \lambda = \frac{2\pi c}{\omega}}$ ("**Slow**" approximation)

[The current changes slowly compared with the time light takes to cross the loop.]

With Approximation (9) for $\mathcal{R}$ and a few trigonometric identities, this gives

$$\cos\left[\omega\left(t - \frac{\mathcal{R}}{c}\right)\right] \approx \cos\left[\omega\left(t - \frac{r}{c}\right)\right] - \frac{\omega b}{c}\sin\theta\,\cos\phi'\,\sin\left[\omega\left(t - \frac{r}{c}\right)\right] \tag{11}$$

and $$\vec{\boldsymbol{A}} \approx \frac{\mu_0 m_0}{4\pi}\left(\frac{\sin\theta}{r}\right)\left\{\frac{1}{r}\cos\left[\omega\left(t - \frac{\mathcal{R}}{c}\right)\right] - \frac{\omega}{c}\sin\left[\omega\left(t - \frac{r}{c}\right)\right]\right\}\hat{\boldsymbol{\phi}} \tag{12}$$

OUTLINE

Distant approximation

Approximation 3: $\boxed{r \gg \lambda = \frac{2\pi c}{\omega}}$ ("**Distant**" approximation)

[Changes are rapid compared with the time light takes to reach the field point.]

This allows us to neglect the $1/r$ term in Eq. (12) for $\vec{\boldsymbol{A}}$, leaving

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}, t) \approx -\frac{\mu_0 m_0}{4\pi} \left(\frac{\sin\theta}{r} \right) \frac{\omega}{c} \sin\left[\omega \left(t - \frac{r}{c} \right) \right] \hat{\boldsymbol{\phi}} \,, \tag{13}$$

of which we take derivatives to find the *fields*: $V = 0$, so

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}, t) = -\frac{\partial \vec{\boldsymbol{A}}}{\partial t} \approx \frac{\mu_0 m_0 \, \omega^2}{4\pi \, c} \left(\frac{\sin\theta}{r} \right) \cos\left[\omega \left(t - \frac{r}{c} \right) \right] \hat{\boldsymbol{\phi}} \qquad \text{and} \tag{14}$$

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}, t) = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}} \approx -\frac{\mu_0 m_0 \, \omega^2}{4\pi \, c^2} \left(\frac{\sin\theta}{r} \right) \cos\left[\omega \left(t - \frac{r}{c} \right) \right] \hat{\boldsymbol{\theta}} \,. \tag{15}$$

OUTLINE

Summary of Magnetic Dipole Radiation

$$\vec{\boldsymbol{E}} = -\frac{\mu_0\, m_0\, \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t - \frac{r}{c}\right)\right]\ \hat{\boldsymbol{\phi}} \tag{16}$$

$$\vec{\boldsymbol{B}} = -\frac{\mu_0\, m_0\, \omega^2}{4\pi c^2}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t - \frac{r}{c}\right)\right]\ \hat{\boldsymbol{\theta}} \tag{17}$$

$$\langle\vec{\boldsymbol{S}}\rangle = \frac{\mu_0\, m_0^2\, \omega^4}{32\pi^2 c^3}\left(\frac{\sin^2\theta}{r^2}\right)\ \hat{\boldsymbol{r}} \tag{18}$$

$$\langle P\rangle = \frac{\mu_0\, m_0^2\, \omega^4}{12\pi c^3} \tag{19}$$

Note: *differs* from electric dipole radiation only by the substitution $p_0 \longrightarrow \dfrac{m_0}{c}$ and the juxtaposition of the *directions* of $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$.

OUTLINE

Electric *vs.* Magnetic Dipole Antennas

The power radiated by magnetic and electric dipoles differs by a factor of

$$\frac{\langle P_{\mathrm{magn}}\rangle}{\langle P_{\mathrm{elec}}\rangle} = \frac{\left(\frac{\mu_0\, m_0^2\, \omega^4}{12\pi c^3}\right)}{\left(\frac{\mu_0\, p_0^2\, \omega^4}{12\pi c}\right)} = \left(\frac{m_0}{p_0 c}\right)^2 \qquad \begin{array}{rl} \text{where} & m_0 = \pi b^2 I_0 \\ \text{and} & p_0 = q_0 d \end{array} . \tag{20}$$

If we imagine the same power supply driving the current I_0 in the magnetic dipole or the oscillating charge in the electric dipole, we can set $I_0 = \omega q_0$.

If we compare dipoles of the "same size" in terms of how far the charge has to move to get to the "other size", we should set $d = \pi b$.

Then $\frac{m_0}{p_0 c} = \frac{b\omega}{c}$. But we just assumed $b \ll \frac{c}{\omega}$ (the "**slow**" appoximation)

so this ratio is $\ll 1$, and it's *squared* in the above formula! Thus we expect *magnetic* dipole radiation to be *extremely puny* compared with *electric* dipole radiation. *But*. . .

OUTLINE

Size Matters!

. . . How valid is the "**slow**" approximation $b \ll \lambda$ in practice?

Suppose your cell phone were using a 10 cm radius magnetic dipole antenna to transmit at 1 GHz; then you would have $b = \lambda/3$, hardly satisfying the "$\ll$" criterion!

In that case the magnetic dipole antenna might not be such a bad idea.

However, I wouldn't want to have to **calculate** its radiation field, because one of our key simplifying assumptions would be incorrect, and I'd have to include more terms. . . .

We make these assumptions so that we *can* calculate something with a manageable amount of effort. But never forget that there are many levels of interesting phenomena accessible with more difficulty!

OUTLINE