UBC Physics 401 Lecture:

DRIVEN CHARGES

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http://musr.physics.ubc.ca/p401/pdf/DrivenChargesPlus.pdf

OUTLINE

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Driving Free Electrons

At one position in space, a plane EM wave $\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t) = \vec{\boldsymbol{E}}_0 e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}} - \omega t)}$ produces an oscillating electric field $\vec{\boldsymbol{E}}_0 e^{-i\omega t}$. If at that position there is a free particle of mass m and charge q, NEWTON'S SECOND LAW says

$$m\frac{d\vec{\boldsymbol{v}}}{dt} = q\vec{\boldsymbol{E}}_0 e^{-i\omega t} - m\gamma\vec{\boldsymbol{v}} \,, \tag{1}$$

where γ is a "viscous" damping rate in s^{-1}. [This is plausible, but difficult to calculate from first principles.] Plugging in a trial steady-state solution of the form $\vec{\boldsymbol{v}}(t) = \vec{\boldsymbol{v}}_0 e^{-i\omega t}$, we find that this will work if

$$\vec{\boldsymbol{v}}_0 = \frac{q\vec{\boldsymbol{E}}_0}{m(\gamma - i\omega)} \,. \tag{2}$$

Conundrum: We are ignoring the effect of the concomitant magnetic field $\vec{\boldsymbol{B}} \perp \vec{\boldsymbol{v}} \parallel \vec{\boldsymbol{E}}$, which will deflect the particles' paths if they develop a significant speed; with strong damping this effect is suppressed, but what happens if $\gamma \to 0$?

If there are N such particles per unit volume in that region, they form a *current density* $\vec{\boldsymbol{J}} = Nq\vec{\boldsymbol{v}}$. Thus Eq. (2) is equivalent to

$$\vec{\boldsymbol{J}} = \frac{q^2 N}{m(\gamma - i\omega)} \vec{\boldsymbol{E}} \, , \tag{3}$$

which is just OHM'S LAW, $\vec{\boldsymbol{J}} = \sigma \vec{\boldsymbol{E}}$, if we define a **complex conductivity**

$$\sigma = \frac{q^2 N}{m(\gamma - i\omega)} \, . \tag{4}$$

This is the *frequency-dependent* version of DRUDE THEORY.

For a *good conductor* like **copper**, $\gamma \sim 10^{13}$ s^{-1}, ensuring that σ is *pure real* up to frequencies $\omega \sim 10^{13}$ s^{-1}, $i.e.$ in between the microwave and infrared ranges.

However, in a *tenuous plasma* where the charged particles almost never collide, γ vanishes and σ is *pure imaginary* (there are no resistive losses). We will look at this case in some more detail.

EM Waves in a Plasma

In a *thin* plasma ($\gamma \to 0$) we can write Eq. (4) as

$$\sigma = i\epsilon_0 \frac{\omega_p^2}{\omega} \qquad \text{where} \qquad \omega_p^2 \equiv \frac{Nq^2}{m\epsilon_0} \equiv (\textbf{plasma frequency})^2. \tag{5}$$

Assuming $\epsilon \approx \epsilon_0$ and $\mu \approx \mu_0$ (as seems reasonable for a near vacuum) this gives a complex wavevector (squared) of

$$\tilde{k}^2 = \mu\epsilon\omega^2 \left(1 + \frac{i\sigma}{\epsilon\omega}\right) \approx \frac{1}{c^2}\left(\omega^2 - \omega_p^2\right) \tag{6}$$

Thus for $\omega < \omega_p$ there is no propagating wave in the plasma, and (since there is also no dissipation mechanism) the plasma is a **perfect reflector** for **low** frequencies.

Equation (6) predicts that for $\omega > \omega_p$ there is no κ (infinite "skin depth") — *i.e.* the wave propagates freely.

But its *propagation speeds* are bizarre:

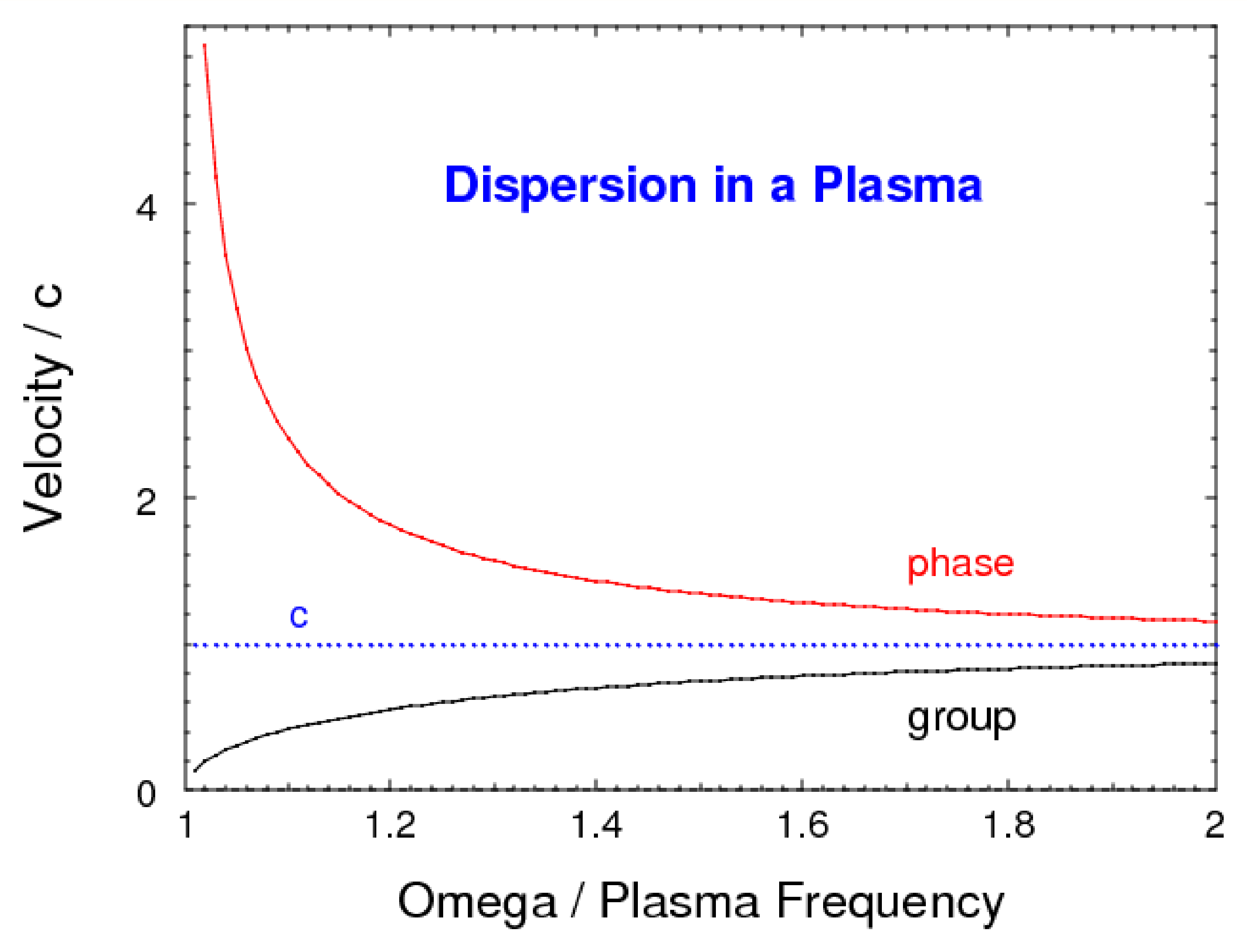
The **phase** or "**wave**" velocity $v_{\rm ph}$ and the **group** velocity v_g behave as

$$v_{\rm ph} \equiv \frac{\omega}{k} = c\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{-1/2} > c\,; \qquad v_g \equiv \frac{d\omega}{dk} = c\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2} < c \qquad (7)$$

respectively; $v_{\rm ph}$ *diverges* whereas $v_g \to 0$ as $\omega \to \omega_p$ from above.

Remember, no actual information or energy moves at $v_{\rm ph}$. However, it seems peculiar to have even a *phase* velocity becoming infinite! This is analogous to the apparent speed of a bright spot on the wall illuminated by a laser pointer: if I rotate the source fast enough, the spot will "move" faster than c.

Phase and Group Velocities



Plot of Eqs. (7).

Looking at it another way

Going back to Eq. (1), and assuming no damping ($\gamma \to 0$), we could just as well solve for the steady-state *position* of the charged particle as a function of time,

$$\vec{\boldsymbol{x}} = -\frac{q\vec{\boldsymbol{E}}_0}{m\omega^2}\vec{\boldsymbol{E}} \ . \qquad (8)$$

where we measure $\vec{\boldsymbol{x}}$ from the equilibrium position of the particle. Displacing a charge produces an electric dipole moment $\vec{\boldsymbol{p}} = q\vec{\boldsymbol{x}}$. If there are N such dipoles per unit volume they form a *polarization* $\vec{\boldsymbol{P}} = Nq\vec{\boldsymbol{x}}$. Thus Eq. (8) is equivalent to

$$\vec{\boldsymbol{P}} \equiv \chi_e \vec{\boldsymbol{E}} = -\frac{q^2 N}{m\omega^2}\vec{\boldsymbol{E}} \qquad \text{or} \qquad \epsilon = \epsilon_0 \left(1 + \frac{\omega_p^2}{\omega^2} \right) \ . \qquad (9)$$

(I think I lost track of a sign in there somewhere!)

You can think in terms of a frequency-dependent dielectric constant *OR* in terms of an imaginary conductivity; but **don't try both at once!**

(See what happens to $\tilde{k}^2$ in Eq. (6) if you do.)

The Ionosphere

What Is It?

- Ionization of upper atmosphere by the Sun's ultraviolet
- Variation with Altitude A
  - Variation of density with A
  - Intensity of sunlight *vs.* A
  - Layers
  - Diurnal variations
  - Plasma Frequency $\omega_p(A)$

Reflection of Radio Waves

- Single bounce
- Reflections off the ground
- The atmosphere as a waveguide

Driving Bound Electrons

If our charged particle ($e.g.$ an electron) is *bound* to some fixed equilibrium position (a heavy nucleus will do) and driven by an oscillatory $\vec{\boldsymbol{E}}$,

$$m\frac{d\vec{\boldsymbol{x}}}{dt} = q\vec{\boldsymbol{E}}_0 e^{-i\omega t} - m\gamma\frac{d\vec{\boldsymbol{x}}}{dt} - m\omega_0^2\frac{d^2\vec{\boldsymbol{x}}}{dt^2} \tag{10}$$

for a damping rate γ and a resonant frequency ω_0 of the particle in its local potential well [almost always a good approximation for small displacements]. Plugging in $\vec{\boldsymbol{x}}(t) = \vec{\boldsymbol{x}}_0 e^{-i\omega t}$, we get

$$\vec{\boldsymbol{x}}_0 = \frac{q/m}{\omega_0^2 - \omega^2 - i\gamma\omega}\vec{\boldsymbol{E}}_0 \; . \tag{11}$$

Again we get a dipole moment $\vec{\boldsymbol{p}} = q\vec{\boldsymbol{x}}$, and if there are N such dipoles per unit volume, the *polarization* is $\vec{\boldsymbol{P}} = Nq\vec{\boldsymbol{x}}$ so Eq. (11) defines a **complex susceptibility**

$$\tilde{\chi}_e = \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega} \; . \tag{12}$$

If we have an assortment of species, each with its own characteristic charge q_j, mass m_j, number density N_j, plasma frequency $\omega_{p_j} = N_j q_j^2/m_j\epsilon_0$, resonant frequency ω_j and damping rate γ_j, then

$$\tilde{\epsilon} = \epsilon_0 \left(1 + \sum_j \frac{\omega_{p_j}^2}{\omega_j^2 - \omega^2 - i\gamma_j\omega} \right) \; . \tag{13}$$

Recalling that $\tilde{k} = \omega\sqrt{\mu\tilde{\epsilon}} = k + i\kappa$, if we assume $\mu \approx \mu_0$ and $|\tilde{\chi}_e| \ll 1$
so that $\sqrt{\mu\tilde{\epsilon}} = \sqrt{\epsilon_0\mu_0}(1 + \chi\ e)^{1/2} \approx \frac{1}{c}\left(1 + \frac{1}{2}\chi_e\right)$,

$$\tilde{k} \approx k_0 \left(1 + \frac{1}{2} \sum_j \frac{\omega_{p_j}^2}{\omega_j^2 - \omega^2 - i\gamma_j\omega} \right) \tag{14}$$

where $k_0 \equiv \omega/c$ is the wavevector of a free plane wave at that frequency.

. . . after some algebra. . .

We get an **index of refraction**

$$n \equiv \frac{ck}{\omega} \approx 1 + \frac{1}{2}\sum_j \omega_{p_j}^2 \left(\frac{\omega_j^2 - \omega^2}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \right) \tag{15}$$

describing the *phase velocity* of the wave

and an **absorption coefficient**

$$\alpha \equiv 2\kappa \approx \frac{1}{c}\sum_j \omega_{p_j}^2 \left(\frac{\gamma \omega^2}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \right) \tag{16}$$

describing the *attenuation of energy* in the wave.