RADIATION

http://musr.physics.ubc.ca/p401/pdf/RadiationPlus.pdf

by

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OUTLINE

The Retarded Potential Revisited

- Far Field approximation
- Slow approximation
- The Radiation Fields
- Radiated Power
- Relativistic Radiation
- Angular Distributions

The Retarded Potential Revisited

$$A^\mu(\vec{\boldsymbol{r}}, t) = \frac{\mu_0}{4\pi} \iiint \frac{J^\mu(\vec{\boldsymbol{r}}', t_r) d\tau'}{\mathcal{R}} \tag{1}$$

where $t_r \equiv t - \frac{\mathcal{R}}{c}$ is the **retarded time** and $\vec{\boldsymbol{\mathcal{R}}} \equiv \vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'$ so that

$$\mathcal{R} = \sqrt{r^2 + r'^2 - 2\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{r}}'} = r \left[1 + \left(\frac{r'}{r} \right)^2 - 2 \left(\frac{\hat{\boldsymbol{r}} \cdot \vec{\boldsymbol{r}}'}{r} \right) \right]^{1/2} .$$

Equation (1) is *exact*, but (as we have seen) difficult to use. Let's see if we can simplify it using some *approximations*.

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Far Field approximation

Approximation 1: $\boxed{r'^2 \ll r^2}$ ("**Far Field** approximation")

$$\text{Then} \qquad \mathcal{R} \approx r\left[1 - 2\left(\frac{\hat{\boldsymbol{r}}\cdot\vec{\boldsymbol{r}}'}{r}\right)\right]^{1/2} \approx r\left[1 - \left(\frac{\hat{\boldsymbol{r}}\cdot\vec{\boldsymbol{r}}'}{r}\right)\right] \tag{2}$$

$$\text{and} \qquad \frac{1}{\mathcal{R}} \approx \frac{1}{r}\left[1 - 2\left(\frac{\hat{\boldsymbol{r}}\cdot\vec{\boldsymbol{r}}'}{r}\right)\right]^{-1/2} \approx \frac{1}{r}\left[1 + \left(\frac{\hat{\boldsymbol{r}}\cdot\vec{\boldsymbol{r}}'}{r}\right)\right] \tag{3}$$

so $t_r \approx t - \frac{r}{c} + \frac{\hat{\boldsymbol{r}}\cdot\vec{\boldsymbol{r}}'}{c}$. We then expand $J^\mu(\vec{\boldsymbol{r}}', t_r)$ as a Taylor series about $t_0 \equiv t - \frac{r}{c}$, the retarded time at the origin:

$$J^\mu(\vec{\boldsymbol{r}}', t_r) \approx J^\mu(\vec{\boldsymbol{r}}', t_0) + \dot{J}^\mu(\vec{\boldsymbol{r}}', t_0)\left(\frac{\hat{\boldsymbol{r}}\cdot\vec{\boldsymbol{r}}'}{c}\right) + \ddot{J}^\mu(\vec{\boldsymbol{r}}', t_0)\left(\frac{\hat{\boldsymbol{r}}\cdot\vec{\boldsymbol{r}}'}{c}\right)^2 + \cdots \tag{4}$$

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Slow approximation

Approximation 2: $\boxed{\; r' \ll \lambda = \frac{2\pi c}{\omega} \;}$ ("**Slow** approximation")

[The source region changes slowly compared with the time light takes to cross it.]

Then we can neglect all but the first time derivative in the Taylor expansion (4)[1] and plug (3) and (4) back into (1) to get

$$A^\mu(\vec{\boldsymbol{r}}, t) \approx \frac{\mu_0}{4\pi\, r} \iiint \left[J^\mu(\vec{\boldsymbol{r}}', t_0) + \dot{J}^\mu\left(\vec{\boldsymbol{r}}', t_0\right) \left(\frac{\hat{\boldsymbol{r}} \cdot \vec{\boldsymbol{r}}'}{c} \right) \right] \left[1 + \left(\frac{\vec{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}}'}{r} \right) \right] d\tau' \; . \qquad (5)$$

Neglecting terms higher than first order in r', we get

$$A^\mu(\vec{\boldsymbol{r}}, t) \approx \frac{\mu_0}{4\pi\, r} \iiint \left\{ J^\mu(\vec{\boldsymbol{r}}', t_0) + J^\mu(\vec{\boldsymbol{r}}', t_0) \left(\frac{\hat{\boldsymbol{r}} \cdot \vec{\boldsymbol{r}}'}{r} \right) + \dot{J}^\mu(\vec{\boldsymbol{r}}', t_0) \left(\frac{\hat{\boldsymbol{r}} \cdot \vec{\boldsymbol{r}}'}{c} \right) \right\} d\tau' \; . \qquad (6)$$

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[1] Griffiths acknowledges that this is not entirely obvious. Me too.

For the zeroth component ($A^0 \equiv V/c$ and $J^0 \equiv c\rho$), the first term in Eq. (6) integrates to the total charge Q at time t_0 (or any other time, since Q is *conserved*):

$$V(\vec{\boldsymbol{r}},t) \approx \frac{1}{4\pi\epsilon_0\, r}\left\{Q + \frac{\hat{\boldsymbol{r}}}{r}\cdot\iiint \rho(\vec{\boldsymbol{r}}',t_0)\,\vec{\boldsymbol{r}}'d\tau' + \frac{\hat{\boldsymbol{r}}}{c}\cdot\frac{\partial}{\partial t}\iiint \rho(\vec{\boldsymbol{r}}',t_0)\vec{\boldsymbol{r}}'\,d\tau'\right\} \tag{7}$$

where $\hat{\boldsymbol{r}}$ and $\frac{\partial}{\partial t}$ have been brought outside the integral since they don't depend on $\vec{\boldsymbol{r}}'$. Both remaining integrals are equal to the overall dipole moment $\vec{\boldsymbol{p}}$ at time t_0, leaving

$$V(\vec{\boldsymbol{r}},t) \approx \frac{1}{4\pi\epsilon_0}\left\{\frac{Q}{r} + \frac{\hat{\boldsymbol{r}}\cdot\vec{\boldsymbol{p}}(t_0)}{r^2} + \frac{\hat{\boldsymbol{r}}\cdot\dot{\boldsymbol{p}}(t_0)}{r\,c}\right\} \tag{8}$$

in which the first and second terms are just the static potentials of the net monopole and dipole moments of the charge distribution. The last term is the main actor!

"Similar arguments" (mercifully omitted) give the vector potential:

$$\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}},t) \approx \frac{\mu_0}{4\pi}\,\frac{\dot{\boldsymbol{p}}(t_0)}{r}\;. \tag{9}$$

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The Radiation Fields

It remains to take the required derivatives to get $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$.

Approximation 3: $\boxed{\text{Discard } \dfrac{1}{r^2} \text{ terms in } \vec{\boldsymbol{E}} \text{ and } \vec{\boldsymbol{B}}.}$ ("**Radiation** approximation")

We do this because $\vec{\boldsymbol{S}} \equiv \vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}/\mu_0$ drops off as r^{-4} for products of such terms, and where we are going (the "*Radiation Zone*") they will be negligible.[2] The results (skipping some relatively straightforward algebra) are:

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}, t) \approx \frac{\mu_0}{4\pi\, r} \left\{ \hat{\boldsymbol{r}} \times \left[\hat{\boldsymbol{r}} \times \ddot{\boldsymbol{p}}(t_0) \right] \right\} \qquad \vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}, t) \approx -\frac{\mu_0}{4\pi\, r\, c} \left[\hat{\boldsymbol{r}} \times \ddot{\boldsymbol{p}}(t_0) \right] , \qquad (10)$$

$$\text{giving} \qquad \vec{\boldsymbol{S}}(\vec{\boldsymbol{r}}, t) \approx \frac{\mu_0}{16\pi^2\, c} \left(\frac{|\hat{\boldsymbol{r}} \times \ddot{\boldsymbol{p}}(t_0)|}{r} \right)^2 \hat{\boldsymbol{r}} \,. \qquad (11)$$

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[2] There can also be cross terms that drop off as r^{-3}, but they (and other features of the "*Near Field Zone*") are painfully complicated and not usually of interest. The radiation fields are hard enough!

Radiated Power

If we integrate Eq. (11) over a sphere at r we obtain the **total radiated power**:

$$P(t) \equiv \iiint \vec{\boldsymbol{S}}(\vec{\boldsymbol{r}}, t) \cdot d\vec{\boldsymbol{a}} \;\approx\; \frac{\mu_0 \, |\ddot{p}(t_0)|^2}{6\pi \, c} \,. \tag{12}$$

where the time at which $\ddot{p}$ is evaluated still has to be earlier than "now" (t) by the time it took the light to travel from the source out to "here" (r).[3]

Let's relate this back to the simple case of an **accelerating point charge** q: since (relative to the origin) $\vec{\boldsymbol{p}} = q\vec{\boldsymbol{r}}'$, it follows that $\ddot{\boldsymbol{p}} = q\ddot{\boldsymbol{r}}' \equiv q\vec{\boldsymbol{a}}$ where $\vec{\boldsymbol{a}}$ is the **acceleration** of the charge. We can therefore write Eq. (12) as

$$P(t) \approx \frac{\mu_0 \, q^2 \, |\vec{\boldsymbol{a}}(t_0)|^2}{6\pi \, c} \qquad \textbf{Larmor formula}. \tag{13}$$

The conclusion that *accelerated charges radiate* is utterly *incompatible with the stability of atoms!* We had to invent *Quantum Mechanics* to fix this problem.

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[3] Good thing we chose a *sphere* to integrate over, eh?

Relativistic Radiation

Equation (13) is the famous **Larmor formula** for radiation by an accelerated charge. It is strictly valid only in the frame where the particle is (or, more accurately, *was*) instantaneously **at rest** at time t_0.

However, by performing the right *Lorentz transformation* into the frame where q has a velocity $\vec{\boldsymbol{v}}$, one can obtain **Liénard's generalization** which is valid even for ultrarelativistic particles:

$$P(t) \approx \frac{\mu_0 \, q^2 \, \gamma^6}{6\pi \, c} \left(a^2 - \frac{|\vec{\boldsymbol{v}} \times \vec{\boldsymbol{a}}|^2}{c} \right) \qquad \textbf{Liénard's generalization}. \qquad (14)$$

Here we must still evaluate $\vec{\boldsymbol{a}}$, $\vec{\boldsymbol{v}}$ and $\gamma \equiv \dfrac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ at the retarded time t_0,

but only if we are being rigourous about what we mean by "now".

Relativistic motion of q distorts the *angular distribution* of the radiated power from the static "donut" shape to one in which the "sides of the donut" are swept forward.

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Angular Distributions

Equation (11) can be rewritten in terms of the acceleration $\vec{\boldsymbol{a}}$ as

$$\vec{\boldsymbol{S}}(\vec{\boldsymbol{r}},t) \approx \frac{\mu_0 q^2}{16\pi^2 c} \left(\frac{|\hat{\boldsymbol{r}} \times \vec{\boldsymbol{a}}(t_0)|}{r} \right)^2 \hat{\boldsymbol{r}} \; . \tag{15}$$

Applying Lorentz transformations to this distribution gives

$$\text{for} \quad \vec{\boldsymbol{v}} \parallel \vec{\boldsymbol{a}} : \qquad \frac{dP}{d\Omega} \approx \frac{\mu_0 q^2 a^2 \gamma^6}{16\pi^2 c} \, \frac{\sin^2\theta}{[1-(v/c)\cos\theta]^5} \tag{16}$$

where θ is measured relative to the direction of motion, and

$$\text{for} \quad \vec{\boldsymbol{v}} \perp \vec{\boldsymbol{a}} : \qquad \frac{dP}{d\Omega} \approx \frac{\mu_0 q^2 a^2 \gamma^6}{16\pi^2 c} \, \frac{[(v/c)-\cos\theta]^2}{[1-(v/c)\cos\theta]^5} \tag{17}$$

where θ is the angle between $\vec{\boldsymbol{a}}(t_r)$ and $\vec{\boldsymbol{v}}(t_r)$.

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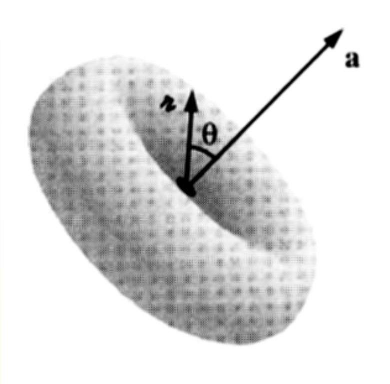
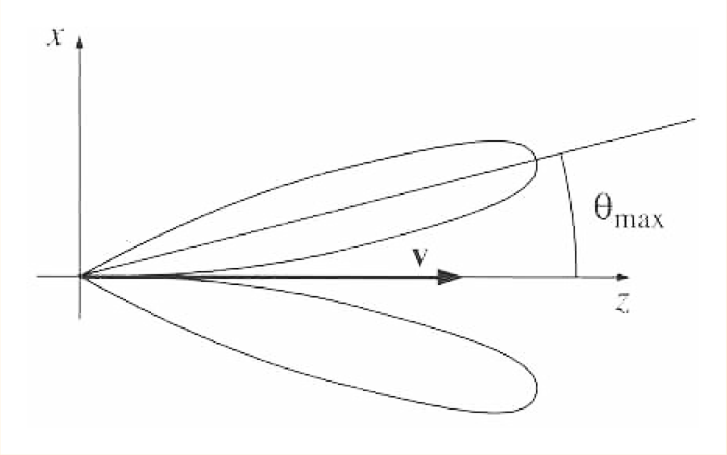
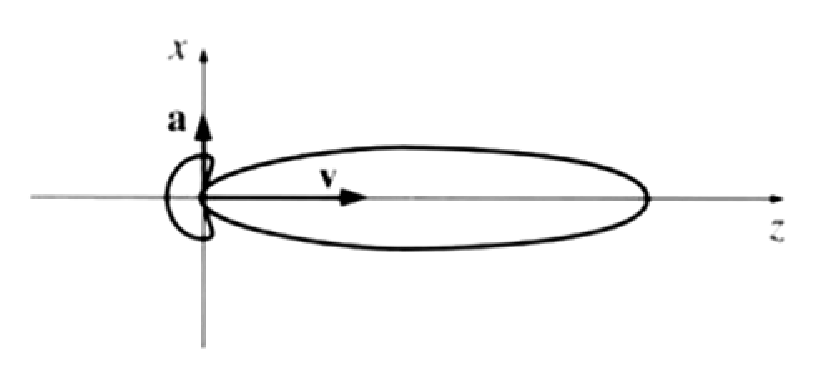
Equation (11) describes the distribution of radiation from an accelerated charge with instantaneous velocity $\vec{v} = 0$:



Equation (16) describes the distribution of **Bremsstrahlung** radiation:



Equation (17) describes the distribution of **Synchrotron** radiation:



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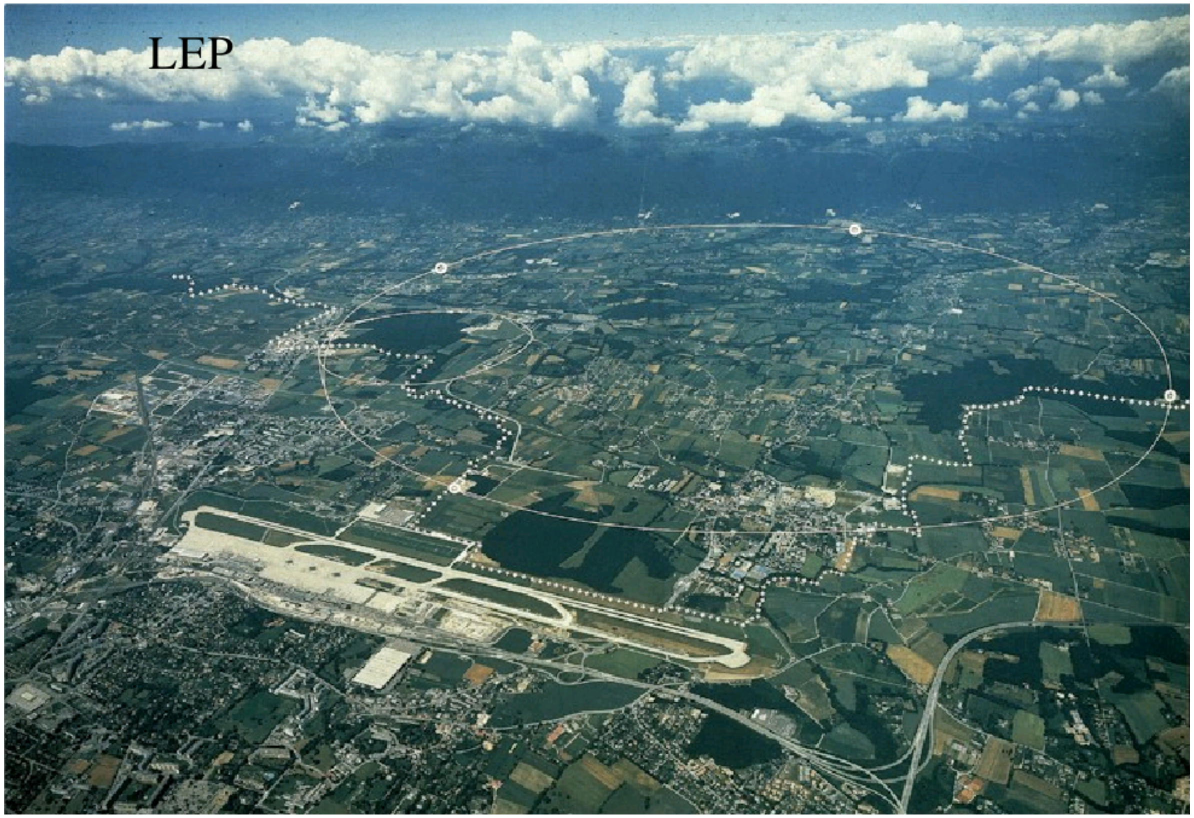
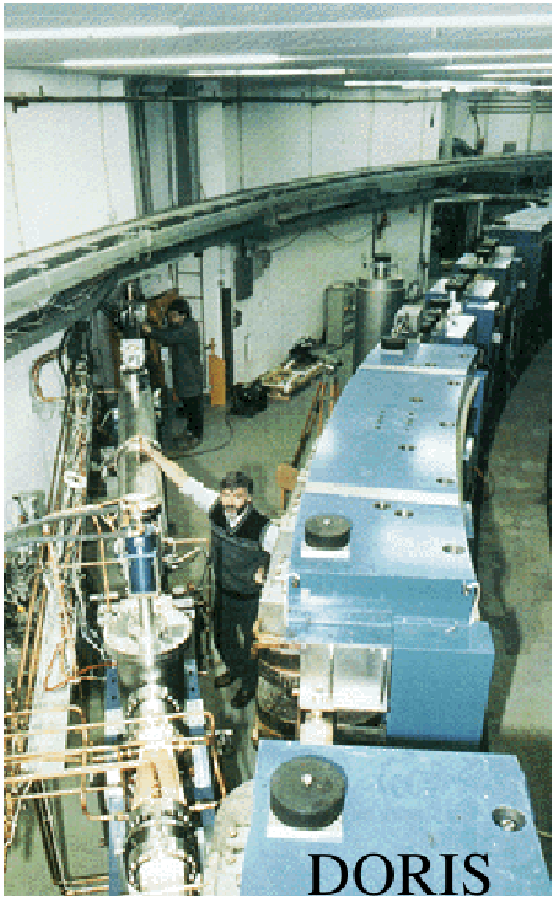
Synchrotron radiation losses were a limiting factor for the Large Electron-Positron (LEP) synchrotron at CERN, which collided e^+ with e^- beams at energies of 90 Gev, and was the largest synchrotron accelerator in the world. The main ring tunnel has a circumference of 26.67 km. It now houses the Large Hadron Collider (LHC) which is scheduled to produce colliding beams of 7 TeV protons starting in 2007. (Synchrotron radiation is much less of a problem for the heavy protons.)



The DORIS synchrotron at the DESY laboratory in Germany uses synchrotron radiation from 4.5 Gev electrons as a sensitive probe of biological and solid state materials.



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The Canadian Light Source (CLS) synchrotron in Saskatoon, SK:

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