UBC Physics 401 Lecture:

RETARDED POTENTIALS

Jess H. Brewer

jess@physics.ubc.ca

http://musr.physics.ubc.ca/p401/pdf/RetardedPlus.pdf

OUTLINE

Maxwell's Equations for the Potential

In the LORENTZ GAUGE $\left(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}} = -\frac{1}{c^2}\frac{\partial V}{\partial t}\right)$, MAXWELL'S EQUATIONS read

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) A^\mu = -\mu_0 J^\mu \qquad \text{where} \qquad A^0 \equiv \frac{V}{c} \quad \text{and} \quad J^0 \equiv c\rho\,. \tag{1}$$

We have seen that for **static** potentials this equation is satisfied by

$$A^\mu(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi}\iiint \frac{J^\mu(\vec{\boldsymbol{r}}')d\tau'}{\mathcal{R}} \qquad \text{where} \qquad \vec{\boldsymbol{\mathcal{R}}} \equiv \vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'\,. \tag{2}$$

(Since LaTeX lacks a nice lowercase script "r", I use $\vec{\boldsymbol{\mathcal{R}}} \equiv \vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'$ in its place.)

Remember, $\vec{\boldsymbol{r}}$ is the position (relative to the origin of coordinates) where we are calculating the EM fields (the "field point") and $\vec{\boldsymbol{r}}'$ is the position (relative to the same origin) of an infinitesimal volume element $d\tau'$ in the source region (the "source point").

OUTLINE

Retarded Time

We can't just use the same formula (2) for the potential when the 4-current is changing with time, because that would imply instantaneous action at a distance — the field point would have to "know" what the source point is doing without any delay due to the finite propagation speed of EM "news".

Instead, we must calculate A^μ from the charge and current distributions at an *earlier* time — earlier by the time it took the "news" to get from the source point to the field point at the speed of light:

$$A^\mu(\vec{\boldsymbol{r}}, t) = \frac{\mu_0}{4\pi} \iiint \frac{J^\mu(\vec{\boldsymbol{r}}', t_r) d\tau'}{\mathcal{R}} \tag{3}$$

where $\boxed{t_r \equiv t - \frac{\mathcal{R}}{c}}$ is the **retarded time**.[1]

OUTLINE

[1] MAXWELL'S EQUATIONS (1) are invariant under both PARITY (inversion of all spatial coordinates) and TIME REVERSAL. Thus an *acausal* version of Eq. (3) using the **advanced time** $t_a \equiv t + \frac{\mathcal{R}}{c}$ would satisfy (1) just as well. For more details see the **Feynman-Wheeler model**.

Spatial Derivatives

Combinations of $\vec{\nabla} \equiv \hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z}$ govern spatial derivatives with respect to the field point $\vec{\boldsymbol{r}} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$: make sure you remember the derivations of

$$\vec{\nabla} r = \hat{\boldsymbol{r}} \qquad \vec{\nabla} \mathcal{R} = \hat{\boldsymbol{\mathcal{R}}} \qquad \vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\hat{\boldsymbol{r}}}{r^2} \qquad \vec{\nabla}\left(\frac{1}{\mathcal{R}}\right) = -\frac{\hat{\boldsymbol{\mathcal{R}}}}{\mathcal{R}^2} \tag{4}$$

$$\vec{\nabla} \cdot \vec{\boldsymbol{r}} = 3 = \vec{\nabla} \cdot \vec{\boldsymbol{\mathcal{R}}} \qquad \vec{\nabla} \cdot \left(\frac{\hat{\boldsymbol{r}}}{r^2}\right) = 4\pi\delta^3(\vec{\boldsymbol{r}}) \qquad \vec{\nabla} \cdot \left(\frac{\hat{\boldsymbol{\mathcal{R}}}}{\mathcal{R}^2}\right) = 4\pi\delta^3(\vec{\boldsymbol{\mathcal{R}}}) \tag{5}$$

$$\vec{\nabla} \cdot \left(\frac{\hat{\boldsymbol{\mathcal{R}}}}{\mathcal{R}}\right) = \frac{1}{\mathcal{R}^2} \qquad \nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\vec{\boldsymbol{r}}) \qquad \nabla^2\left(\frac{1}{\mathcal{R}}\right) = -4\pi\delta^3(\vec{\boldsymbol{\mathcal{R}}})\,. \tag{6}$$

Combinations of $\vec{\nabla}' \equiv \hat{\boldsymbol{x}}'\frac{\partial}{\partial x'} + \hat{\boldsymbol{y}}'\frac{\partial}{\partial y'} + \hat{\boldsymbol{z}}'\frac{\partial}{\partial z'}$ govern spatial derivatives with respect to the source point $\vec{\boldsymbol{r}}' = x'\hat{\boldsymbol{x}}' + y'\hat{\boldsymbol{y}}' + z'\hat{\boldsymbol{z}}'$: these are analogous except that each operation of $\vec{\nabla}'$ on $\mathcal{R}$ picks up an extra minus sign.

OUTLINE

Just to remind you that these simple relations (4-6) are straightforward but not trivial to derive, let me show the algebra for what seems a very simple example (see Problem 1.62 on p. 57):

Using the identity

$$\vec{\boldsymbol{\nabla}} \cdot (f\vec{\boldsymbol{v}}) = f\left(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}}\right) + \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{\nabla}} f\,, \tag{7}$$

$$\text{we get} \qquad \vec{\boldsymbol{\nabla}} \cdot \left(\frac{\hat{\boldsymbol{\mathcal{R}}}}{\mathcal{R}}\right) = \left(\frac{1}{\mathcal{R}}\right) \vec{\boldsymbol{\nabla}} \cdot \hat{\boldsymbol{\mathcal{R}}} + \hat{\boldsymbol{\mathcal{R}}} \cdot \vec{\boldsymbol{\nabla}}\left(\frac{1}{\mathcal{R}}\right)$$

$$\text{where} \qquad \vec{\boldsymbol{\nabla}}\left(\frac{1}{\mathcal{R}}\right) = -\frac{\hat{\boldsymbol{\mathcal{R}}}}{\mathcal{R}^2} \qquad \text{from Eqs. (4) and}$$

$$\vec{\boldsymbol{\nabla}} \cdot \hat{\boldsymbol{\mathcal{R}}} = \vec{\boldsymbol{\nabla}} \cdot \left(\frac{\vec{\boldsymbol{\mathcal{R}}}}{\mathcal{R}}\right) = \frac{1}{\mathcal{R}}\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{\mathcal{R}}} + \vec{\boldsymbol{\mathcal{R}}} \cdot \vec{\boldsymbol{\nabla}}\left(\frac{1}{\mathcal{R}}\right) = \frac{3}{\mathcal{R}} - \vec{\boldsymbol{\mathcal{R}}} \cdot \left(\frac{\hat{\boldsymbol{\mathcal{R}}}}{\mathcal{R}^2}\right) = \frac{3}{\mathcal{R}} - \frac{1}{\mathcal{R}} = \frac{2}{\mathcal{R}}$$

$$\text{giving} \qquad \vec{\boldsymbol{\nabla}} \cdot \left(\frac{\hat{\boldsymbol{\mathcal{R}}}}{\mathcal{R}}\right) = \frac{2}{\mathcal{R}^2} - \frac{1}{\mathcal{R}^2} = \frac{1}{\mathcal{R}^2} \qquad \text{as in Eqs. (6).} \ \checkmark\ \mathcal{QED}$$

OUTLINE

Implicit Time Derivatives

Because we are now evaluating A^μ in terms of the current density $J^\mu(\vec{\boldsymbol{r}}', t_r)$ at the retarded time $t_r \equiv t - \mathcal{R}/c$, derivatives of the latter with respect to $\vec{\boldsymbol{r}}$ (upon which it does *not* depend *explicitly*) mix in the *time* derivative through the *implicit* dependence of t_r on $\vec{\boldsymbol{r}} = \vec{\boldsymbol{\mathcal{R}}} + \vec{\boldsymbol{r}}'$. That is,

$$\vec{\boldsymbol{\nabla}} J^\mu(\vec{\boldsymbol{r}}', t_r) = \left(\frac{\partial\, J^\mu}{\partial t_r}\right) \vec{\boldsymbol{\nabla}} t_r = -\dot{J}^\mu \frac{\hat{\boldsymbol{\mathcal{R}}}}{c} \tag{8}$$

because, for a given $\mathcal{R}$, $\dfrac{\partial\, J^\mu}{\partial t_r} = \dfrac{\partial\, J^\mu}{\partial t} = \dot{J}^\mu$, and $\vec{\boldsymbol{\nabla}} t_r = -\dfrac{1}{c}\vec{\boldsymbol{\nabla}}\mathcal{R} = -\dfrac{\hat{\boldsymbol{\mathcal{R}}}}{c}$.

Thus, for example, if we take the gradient $\vec{\boldsymbol{\nabla}} A^\mu$, the operator comes inside the integral in Eq. (3) (which is over $\vec{\boldsymbol{r}}'$) leaving us to evaluate

$$\vec{\boldsymbol{\nabla}} \left[\frac{J^\mu(\vec{\boldsymbol{r}}', t_r)}{\mathcal{R}}\right] = J^\mu \vec{\boldsymbol{\nabla}} \left(\frac{1}{\mathcal{R}}\right) - \frac{\hat{\boldsymbol{\mathcal{R}}}}{c} \dot{J}^\mu \left(\frac{1}{\mathcal{R}}\right) = -J^\mu \left(\frac{\hat{\boldsymbol{\mathcal{R}}}}{\mathcal{R}^2}\right) - \frac{1}{c}\dot{J}^\mu \left(\frac{\hat{\boldsymbol{\mathcal{R}}}}{\mathcal{R}}\right) \tag{9}$$

where J^μ is to be understood to mean $J^\mu(\vec{\boldsymbol{r}}', t_r)$ in each case.

OUTLINE

Checking the Retarded Potential

If we now take the divergence of Eq. (9) we can check whether the RETARDED POTENTIAL (3) really satisfies MAXWELL'S EQUATIONS (1). For this we need the identity (7) and several of Eqs. (4-6).

$$\nabla^2 \left[\frac{J^\mu}{\mathcal{R}} \right] = \vec{\boldsymbol{\nabla}} \cdot \left[-J^\mu \left(\frac{\hat{\boldsymbol{\mathcal{R}}}}{\mathcal{R}^2} \right) - \frac{1}{c} \dot{J}^\mu \left(\frac{\hat{\boldsymbol{\mathcal{R}}}}{\mathcal{R}} \right) \right]$$

$$= -J^\mu \vec{\boldsymbol{\nabla}} \cdot \left(\frac{\hat{\boldsymbol{\mathcal{R}}}}{\mathcal{R}^2} \right) - \left(\frac{\hat{\boldsymbol{\mathcal{R}}}}{\mathcal{R}^2} \right) \cdot \vec{\boldsymbol{\nabla}} J^\mu - \frac{1}{c} \dot{J}^\mu \vec{\boldsymbol{\nabla}} \cdot \left(\frac{\hat{\boldsymbol{\mathcal{R}}}}{\mathcal{R}} \right) - \frac{1}{c} \left(\vec{\boldsymbol{\nabla}} \dot{J}^\mu \right) \cdot \left(\frac{\hat{\boldsymbol{\mathcal{R}}}}{\mathcal{R}} \right)$$

$$= 4\pi J^\mu \delta^3(\vec{\boldsymbol{\mathcal{R}}}) + \frac{1}{c} \dot{J}^\mu \left(\frac{\hat{\boldsymbol{\mathcal{R}}}}{\mathcal{R}^2} \right) \cdot \hat{\boldsymbol{\mathcal{R}}} - \frac{1}{c} \frac{\dot{J}^\mu}{\mathcal{R}^2} + \frac{1}{c^2} \ddot{J}^\mu \hat{\boldsymbol{\mathcal{R}}} \cdot \left(\frac{\hat{\boldsymbol{\mathcal{R}}}}{\mathcal{R}} \right)$$

$$\text{or} \qquad \nabla^2 \left[\frac{J^\mu}{\mathcal{R}} \right] = 4\pi J^\mu \delta^3(\vec{\boldsymbol{\mathcal{R}}}) + \frac{1}{c^2} \frac{\ddot{J}^\mu}{\mathcal{R}} \tag{10}$$

Now to put this integrand back into the Laplacian of Eq. (3) . . .

OUTLINE

$$\nabla^2 A^\mu = \frac{\mu_0}{4\pi} \iiint d\tau' \left\{ \nabla^2 \left[\frac{J^\mu}{\mathcal{R}} \right] \right\} \;=\; \frac{\mu_0}{4\pi} \iiint d\tau' \left\{ 4\pi J^\mu \delta^3(\vec{\boldsymbol{\mathcal{R}}}) + \frac{1}{c^2} \frac{\ddot{J}^\mu}{\mathcal{R}} \right\}$$

$$= \mu_0 J^\mu + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \left(\frac{\mu_0}{4\pi} \iiint \frac{J^\mu d\tau'}{\mathcal{R}} \right)$$

$$\text{or} \qquad \left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) A^\mu = \mu_0 J^\mu \qquad \checkmark \; \mathcal{QED}$$

The RETARDED POTENTIAL does indeed satisfy MAXWELL'S EQUATIONS (1) in the LORENTZ GAUGE.

We can therefore use it to find the EM fields generated by any sort of time-dependent charge and current distribution.

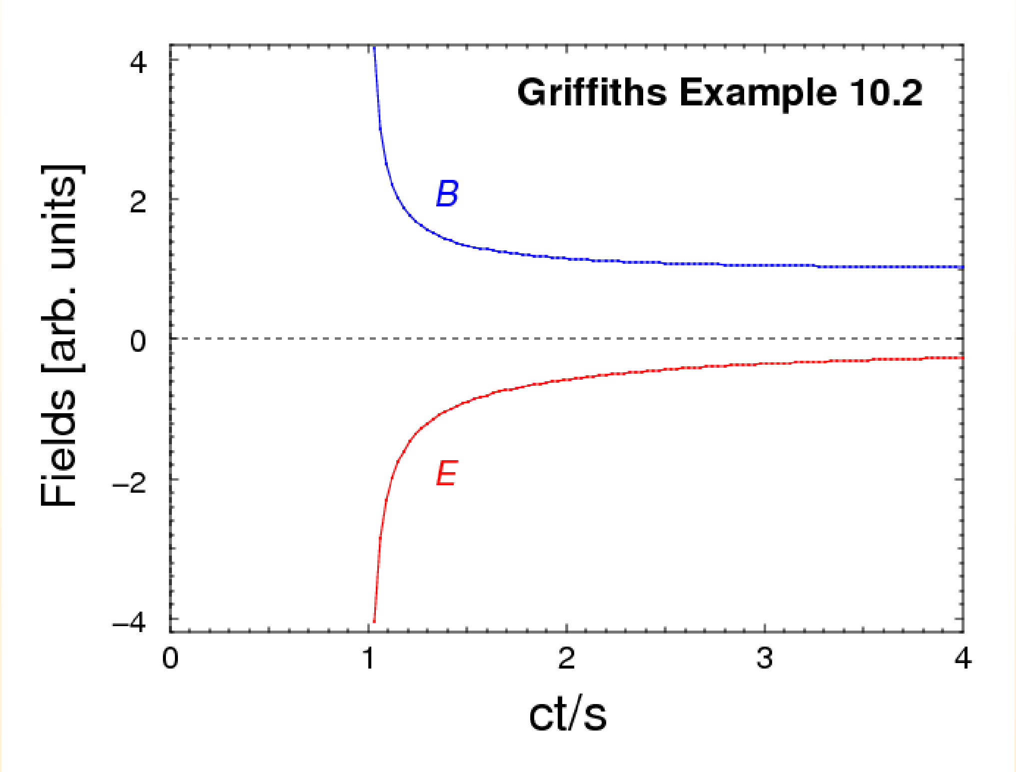
OUTLINE

Using the Retarded Potential

You should be personally convinced of the rigourous validity of the general formula (3) for the RETARDED POTENTIAL, but (apart from computer software) one rarely attempts to apply it directly to the solution of real-life problems.

Usually we have some idealized geometry like an infinite straight wire and a simplified time dependence like turning on a steady current I_0 abruptly at $t = 0$ (Example 10.2 in Griffiths).[2]

The result looks like this:



OUTLINE

[2] One should be cautious about using "infinite" objects, since the derivation of Eq. (2) involved an assumption that $\vec{J} \to 0$ at infinity (see p. 235 in Griffiths).

It would also be difficult to arrange such an instantaneous onset of current everywhere in an infinite straight wire. While the charges that go into the wire at one end are not the same charges that start coming out at the other end, it takes time for the "EM News" (about whatever is driving the current) to reach the charges down the line.

Jefimenko's Equations

It was stated earlier that only the *potential* A^μ can be expressed simply in terms of the retarded time; the *fields*, being local *derivatives* of the potential, are even more intricate to describe explicitly. But it can be done, by simply taking the required derivatives of the general expression (3):

$$\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}} V \equiv -c\vec{\boldsymbol{\nabla}} A^0 \qquad \text{and} \qquad \vec{\boldsymbol{B}} = -\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}} \,, \tag{11}$$

giving (after a bit of algebra, see pp. 427-8)

$$\vec{\boldsymbol{E}} = \frac{1}{4\pi\epsilon_0} \iiint d\tau' \left[\frac{\rho(\vec{\boldsymbol{r}}', t_r)}{\mathcal{R}^2} \hat{\boldsymbol{\mathcal{R}}} + \frac{\dot{\rho}(\vec{\boldsymbol{r}}', t_r)}{c\mathcal{R}} \hat{\boldsymbol{\mathcal{R}}} - \frac{\dot{\boldsymbol{J}}(\vec{\boldsymbol{r}}', t_r)}{c^2\mathcal{R}} \right] \tag{12}$$

$$\text{and} \qquad \vec{\boldsymbol{B}} = \frac{\mu_0}{4\pi} \iiint d\tau' \left[\frac{\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}}', t_r)}{\mathcal{R}^2} + \frac{\dot{\boldsymbol{J}}(\vec{\boldsymbol{r}}', t_r)}{c\mathcal{R}} \right] \,. \tag{13}$$

As Griffiths points out, it is almost always earier to calculate A^μ from (3) first and take the derivatives (11) as needed.

OUTLINE

Liénard-Wiechert Potentials

Suppose J^μ consists of a single point charge q in motion, following a *trajectory* $\vec{\boldsymbol{w}}(t)$. Then $\vec{\boldsymbol{\mathcal{R}}}(t) = \vec{\boldsymbol{r}} - \vec{\boldsymbol{w}}(t)$ and J^μ must be expressed in terms of a rather exotic δ-function. I have attempted to find a more intuitive explanation for the result than you find in the various textbooks, and failed. Griffiths is actually one of the less formal derivations; for a truly deep understanding you should read them all, for this is probably *the* most conceptually challenging topic in E&M. I must reluctantly resort to expressing the result in compact form:

$$A^\mu(\vec{\boldsymbol{r}}, t) = \frac{\mu_0 q}{4\pi} \left[\frac{\beta^\mu}{\mathcal{R}(1 - \vec{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{\mathcal{R}}})} \right]_{\text{ret}} \qquad \text{where} \qquad \beta^\mu = \left\{ 1, \frac{\dot{\boldsymbol{w}}}{c} \right\} \tag{14}$$

is the "4-velocity" (/c) and $[\cdots]_{\text{ret}}$ means that the quantities in the square brackets (including β^μ) are to be evaluated at the retarded time $t_r = t - \mathcal{R}/c$.

This is not as simple as it sounds, of course, since (for instance) we have to evaluate $\mathcal{R}$ at a time that *depends on* $\mathcal{R}$.

OUTLINE

Fields of a Moving Point Charge

What we usually *want* is $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ due to the moving point charge. In principle all we have to do is take the requisite derivatives of A^μ:

$$\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}} V - \frac{\partial \vec{\boldsymbol{A}}}{\partial t} \qquad \text{and} \qquad \vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}$$

However, as Griffiths notes, taking these derivatives under the $[\cdots]_{\rm ret}$ evaluation in Eq. (14) is "rough going . . . but the answer is worth the effort."

Here (sorry, I can't do better than Griffiths!) is the answer:

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}, t) = \frac{q}{4\pi\epsilon_0} \frac{\mathcal{R}}{(\vec{\boldsymbol{\mathcal{R}}} \cdot \vec{\boldsymbol{u}})^3} \left[(c^2 - v^2)\vec{\boldsymbol{u}} + \vec{\boldsymbol{\mathcal{R}}} \times (\vec{\boldsymbol{u}} \times \vec{\boldsymbol{a}}) \right] \tag{15}$$

$$\text{where} \qquad \vec{\boldsymbol{u}} \equiv c\hat{\boldsymbol{\mathcal{R}}} - \vec{\boldsymbol{v}} \qquad \text{and} \qquad \vec{\boldsymbol{v}} \equiv \dot{\boldsymbol{w}} \tag{16}$$

We can always get $\vec{\boldsymbol{B}}$ from $\vec{\boldsymbol{E}}$ because $\vec{\boldsymbol{B}} = \frac{1}{c}\hat{\boldsymbol{\mathcal{R}}} \times \vec{\boldsymbol{E}}$, which tells us that $\vec{\boldsymbol{B}} \perp \vec{\boldsymbol{E}}$!

OUTLINE

Equation (15) can be written

$$\vec{\boldsymbol{E}} = \vec{\boldsymbol{E}}_v + \vec{\boldsymbol{E}}_a \qquad\qquad \text{where} \tag{17}$$

$$\vec{\boldsymbol{E}}_v \equiv \frac{q}{4\pi\epsilon_0} \frac{\mathcal{R}}{(\vec{\boldsymbol{\mathcal{R}}} \cdot \vec{\mathfrak{B}})^3} (1 - \beta^2)\vec{\mathfrak{B}} \tag{18}$$

$$\vec{\boldsymbol{E}}_a \equiv \frac{q}{4\pi\epsilon_0} \frac{1}{c} \frac{\mathcal{R}}{(\vec{\boldsymbol{\mathcal{R}}} \cdot \vec{\mathfrak{B}})^3} \left[\vec{\boldsymbol{\mathcal{R}}} \times (\vec{\mathfrak{B}} \times \vec{\boldsymbol{\alpha}})\right] \tag{19}$$

$$\vec{\boldsymbol{\beta}} \equiv \frac{\dot{\boldsymbol{w}}}{c}, \qquad \vec{\mathfrak{B}} \equiv \hat{\boldsymbol{\mathcal{R}}} - \vec{\boldsymbol{\beta}} \qquad \text{and} \qquad \vec{\boldsymbol{\alpha}} \equiv \frac{\ddot{\boldsymbol{w}}}{c}\,. \tag{20}$$

(I have "compacted" the notation a little more than Griffiths. Does it help?) **Note**: *everything still must be evaluated at the retarded time!*

Note that the **velocity field** $\vec{\boldsymbol{E}}_v$ depends only on the *velocity* of the charge and drops off as $\mathcal{R}^{-2}$ (like the static field), while the **acceleration field** $\vec{\boldsymbol{E}}_a$ depends on both the velocity and the *acceleration* of the charge and drops off as $\mathcal{R}^{-1}$ (like an outgoing spherical wave). The latter dominates at large distances from the source, and is (not surprisingly) also called the **radiation field**.

OUTLINE

Point Charge with Constant Velocity

For a point charge with constant velocity $\vec{v}$, we have[3] $\vec{a} = 0$ so $\vec{\alpha} = 0$ and if we take the origin to be at the position of the particle at $t = 0$, $\vec{w} = \vec{v}t$. This implies $\vec{\mathcal{R}}(t_r) = \vec{r} - (t - \mathcal{R}/c)\vec{v}$, which is still a transcendental equation, since $\vec{\mathcal{R}}(t_r)$ is a function of itself. However, we can rearrange $t_r = t - \mathcal{R}/c$ to get $\mathcal{R} = c(t - t_r)$. Thus $\mathcal{R}\vec{\mathfrak{B}} = \vec{\mathcal{R}} - \mathcal{R}\vec{\beta} = (\vec{r} - \vec{v}t_r) - (t - t_r)\vec{v} = \vec{r} - \vec{v}t$, liberating one term from "retardedness"! Meanwhile[4] $\vec{\mathcal{R}} \cdot \vec{\mathfrak{B}} = \mathcal{R} - \vec{\mathcal{R}} \cdot \vec{\beta} = \sqrt{(ct - \vec{r} \cdot \vec{\beta})^2 + (1 - \beta^2)(r^2 - c^2t^2)}$, which can be written[5] as $\vec{\mathcal{R}} \cdot \vec{\mathfrak{B}} = R\sqrt{1 - \beta^2 \sin^2 \theta}$ where $\vec{R} \equiv \vec{r} - \vec{v}t$ is the vector from the present location of q to the field point. This is an "*extraordinary* coincidence" and a happy one!

Note that we now have completely eliminated all reference to retarded time. *Yay!* This positions us to write down (at last) a "simple" expression (21) for the fields produced by a moving charge. . . .

OUTLINE

[3] This is Griffiths' Ex. 10.4 on pp. 439-440.
[4] See Griffiths' Ex. 10.3 on pp. 433-434.
[5] See Griffiths' Prob. 10.14 on p. 434.

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}},t) = \left(\frac{q}{4\pi\epsilon_0}\right) \frac{1-\beta^2}{\left(1-\beta^2\sin^2\theta\right)^{3/2}} \left(\frac{\hat{\boldsymbol{R}}}{R^2}\right) \tag{21}$$

where again $\vec{\boldsymbol{R}} = \vec{\boldsymbol{r}} - \vec{\boldsymbol{v}}t$ is the position of the charge q at time t.

As usual $\vec{\boldsymbol{B}} = \frac{1}{c}\left(\hat{\boldsymbol{\mathcal{R}}} \times \vec{\boldsymbol{E}}\right)$, but in this case $\hat{\boldsymbol{\mathcal{R}}} \times \vec{\boldsymbol{E}} = \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}}$, so

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}},t) = \frac{1}{c}\left(\frac{q}{4\pi\epsilon_0}\right) \frac{1-\beta^2}{\left(1-\beta^2\sin^2\theta\right)^{3/2}} \left(\frac{\vec{\boldsymbol{\beta}} \times \hat{\boldsymbol{R}}}{R^2}\right) . \tag{22}$$

These results reduce to the familiar quasistatic approximations in the limit of $\beta \ll 1$, but for highly relativistic charges the $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ fields are strongly "flattened" in the direction of $\vec{\boldsymbol{v}}$.

On to Chapter 11: **Radiation**.

OUTLINE