Calculus in a Nutshell

- **Definition:** The rate of change [slope] of a function at a point is the limiting value of its *average* slope over an interval including that point, as the width of the interval shrinks to zero:
$$\frac{dy}{dx} \equiv \lim_{\Delta x \to 0} \frac{y(x+\Delta x) \; - \; y(x)}{\Delta x}$$
All the remaining Laws and Rules can be proven by algebraic manipulation of this definition.

- **Operator Notation:** The symbol $\frac{d}{dx}$ (read "derivative with respect to x") can be thought of as a mathematical "verb" (called an *operator*) which "operates on" whatever we place to its right. Thus
$$\frac{d}{dx}[y] \equiv \frac{dy}{dx}$$

- **Power Law:** The simplest class of derivatives are those of power-law functions:
$$\frac{d}{dx}[x^p] = p\,x^{p-1}$$
valid for *all* powers p, whether positive, negative, integer, rational, irrational, real, imaginary or complex.

- **Product Law:** The derivative of the product of two functions is *not* the product of their derivatives! Instead,
$$\frac{d}{dx}[f(x)\cdot g(x)] = \frac{df}{dx}\cdot g(x) \; + \; f(x)\cdot\frac{dg}{dx}$$

- **Constant times a Function:** The *Product Law* gives
$$\frac{d}{dx}[a\cdot y(x)] = a\cdot\frac{dy}{dx}$$
where a is a constant (*i.e.*, not a function of x). This is often referred to as "pulling the constant factor outside the derivative."

- **Function of a Function:** Suppose y is a function of x and x is in turn a function of t.

  Then $$\frac{d}{dt}\{y[x(t)]\} = \frac{dy}{dx}\cdot\frac{dx}{dt}$$ (**Chain Rule**).

- **Exponentials:**
$$\frac{d}{dx}\left[e^{kx}\right] = k\cdot e^{kx}$$
where k is any constant.

- **Natural Logarithms:**
$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

- **Antiderivatives:** We can "solve" *integrals* or "antiderivatives" the same way we "solve" long division problems: by trial and error guessing! Suppose we are given an explicit function $f(x)$ [for example, $f(x) = x^2$] and told that $f(x)$ *is the derivative of* a function $y(x)$ which we would like to know — that is,

$$\text{If} \qquad \frac{dy}{dx} = x^2, \qquad \text{what is} \qquad y(x)\,?$$

Well, we know that

$$\frac{d}{dx}\left[x^3\right] \;=\; 3\,x^2,$$

so we must divide by 3 to get

$$y(x) = \tfrac{1}{3}\,x^3 \;+\; y_0$$

where the constant term y_0 (the value of y when $x = 0$) cannot be determined from the information given — the derivative of any constant is zero, so such an *integral* is always undetermined to within such a *constant of integration.*