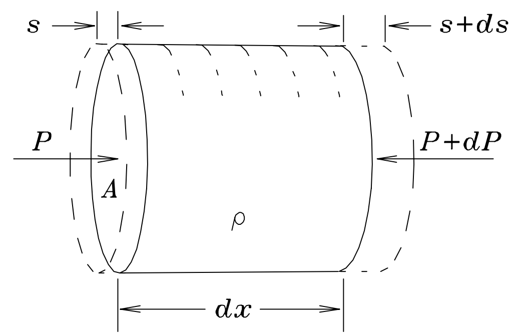
SOUND WAVES



Picture a "snapshot" (holding time t fixed) of a small cylindrical section of an elastic medium, shown at right: the cross-sectional area is A and the length is dx. An excess pressure P (over and above the ambient pressure existing in the medium at equilibrium) is exerted on the left side and a slightly different pressure $P + dP$ on the right. The resulting volume element $dV = Adx$ has a mass $dm = \rho dV = \rho A dx$, where ρ is the mass density of the medium. If we choose the positive x direction to the right, the net force acting on dm in the x direction is $dF_x = PA - (P + dP)A = -AdP$.

Now let s denote the *displacement* of particles of the medium from their equilibrium positions. This may also differ between one end of the cylindrical element and the other: s on the left *vs.* $s + ds$ on the right. We assume the displacements to be in the x direction but *very small* compared to dx, which is itself no great shakes.[1]

The *fractional change in volume* dV/V of the cylinder due to the *difference* between the displacements at the two ends is

$$\frac{dV}{V} = \frac{A(s + ds) - As}{Adx} = \frac{ds}{dx} = \left(\frac{\partial s}{\partial x}\right)_t \tag{1}$$

where the rightmost expression reminds us explicitly that this description is being constructed around a "snapshot" with t held fixed.

Now, any elastic medium is by definition compressible but "fights back" when compressed ($dV < 0$) by exerting a pressure in the direction of increasing volume. The Bulk Modulus B is a constant characterizing how hard the medium fights back — a sort of 3-dimensional analogue of the spring constant. It is defined by

$$P = -B \frac{dV}{V}. \tag{2}$$

Combining Eqs. (1) and (2) gives

$$P = -B \left(\frac{\partial s}{\partial x}\right)_t \tag{3}$$

so that the *difference* in pressure between the two ends is

$$dP = \left(\frac{\partial P}{\partial x}\right)_t dx = -B \left(\frac{\partial^2 s}{\partial x^2}\right)_t dx. \tag{4}$$

We now use $\sum F_x = m a_x$ on the mass element, giving

$$-AdP = AB \left(\frac{\partial^2 s}{\partial x^2}\right)_t dx = dm\, a_x = \rho A dx \left(\frac{\partial^2 s}{\partial t^2}\right)_x \tag{5}$$

where we have noted that the acceleration of all the particles in the volume element (assuming $ds \ll s$) is just $a_x \equiv (\partial^2 s/\partial t^2)_x$.

[1] Note also that any of s, ds, P or dP can be either positive or negative; we merely illustrate the math using an example in which they are all positive.

If we cancel Adx out of Eq. (5), divide through by B and collect terms, we get

$$\left(\frac{\partial^2 s}{\partial x^2}\right)_t - \frac{\rho}{B}\left(\frac{\partial^2 s}{\partial t^2}\right)_x = 0 \qquad \text{or} \qquad \left(\frac{\partial^2 s}{\partial x^2}\right)_t - \frac{1}{c^2}\left(\frac{\partial^2 s}{\partial t^2}\right)_x = 0 \tag{6}$$

which the acute reader will recognize as the WAVE EQUATION in one dimension (x), provided

$$c = \sqrt{\frac{B}{\rho}} \tag{7}$$

is the velocity of propagation.

The fact that disturbances in an elastic medium obey the WAVE EQUATION *guarantees* that such disturbances will propagate as simple *waves* with phase velocity c given by Eq. (7).