The University of British Columbia

Physics 401 Assignment # 1: REVIEW

SOLUTIONS:

Wed. 04 Jan. 2006 — finish by Wed. 11 Jan.

This first assignment was just for review, to make sure you hadn't forgotten (or could quickly recall) what you learned in PHYS 301/354 (or earlier) about the E&M covered in the first 7 chapters of our textbook: David Griffiths, *"Introduction to Electrodynamics"*.

Almost all you need is on the inside front and back covers of the textbook:

$$\text{Gauss' law(s):} \quad \vec{\nabla}\cdot\vec{D} = \rho_f \quad \text{and} \quad \vec{\nabla}\cdot\vec{B} = 0 \qquad (1)$$

$$\text{Faraday's law:} \quad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad (2)$$

$$\text{Ampère's law:} \quad \vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial\vec{D}}{\partial t} \qquad (3)$$

$$\text{Diverg. thm:} \quad \iiint\left(\vec{\nabla}\cdot\vec{A}\right)d\tau = \oiint \vec{A}\cdot d\vec{a} \qquad (4)$$

$$\text{Stokes' thm:} \quad \iint\left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{a} = \oint \vec{A}\cdot d\vec{\ell} \qquad (5)$$

1. MAXWELL'S EQUATIONS:

(*a*) Starting with Maxwell's equations in differential form, derive Maxwell's equations in integral form. **ANSWER**: Applying (4) to $\vec{D}$ yields $\iiint\left(\vec{\nabla}\cdot\vec{D}\right)d\tau = \oiint\vec{D}\cdot d\vec{a}$. Plug in (1) to give $\boxed{\oiint\vec{E}\cdot d\vec{a} = (1/\epsilon_0)\iiint\rho\, d\tau = Q_{\rm enc}/\epsilon_0}$. Similarly for $\vec{B}$ with $\vec{\nabla}\cdot\vec{B} = 0$ to give $\boxed{\oiint\vec{B}\cdot d\vec{a} = 0}$. We use (5) [also known as the curl theorem] on $\vec{E}$ to give $\iint(\vec{\nabla}\times\vec{E})\cdot d\vec{a} = \oint\vec{E}\cdot d\vec{\ell}$ and plug in (2) to get $\iint\left(-\frac{\partial\vec{B}}{\partial t}\right)\cdot d\vec{a} = -\frac{\partial}{\partial t}\iint\vec{B}\cdot d\vec{a} = -\frac{\partial\Phi_M}{\partial t} = \oint\vec{E}\cdot d\vec{\ell} = \text{emf}_{\rm ind}$ or $\boxed{\text{emf}_{\rm ind} = -\frac{\partial\Phi_M}{\partial t}}$. Finally we apply (5) to $\vec{H}$ to get $\iint(\vec{\nabla}\times\vec{H})\cdot d\vec{a} = \oint\vec{H}\cdot d\vec{\ell}$ and plug in (3) to get $\iint\left(\vec{J}_f + \frac{\partial\vec{D}}{\partial t}\right)\cdot d\vec{a} = \frac{\partial}{\partial t}\iint\vec{D}\cdot d\vec{a} + \iint\vec{J}_f\cdot d\vec{a} \equiv \frac{\partial\Phi_E}{\partial t} + I_{\rm encl} = \oint\vec{H}\cdot d\vec{\ell}$ or $\boxed{\oint\vec{H}\cdot d\vec{\ell} = I_{\rm encl} + \frac{\partial\Phi_E}{\partial t}}$.

(*b*) Starting with Maxwell's generalization of Ampère's Law, $\vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial\vec{D}}{\partial t}$, derive the continuity equation, $\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$, which is the mathematical expression of charge conservation. **ANSWER**: Just take the divergence of both sides of Eq. (3), remembering that the divergence of a curl is always zero, and plug in Gauss' law for $\vec{\nabla}\cdot\vec{D}$ (see above). Immediately we have $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{H}) = 0 = \vec{\nabla}\cdot\vec{J} + \frac{\partial\rho_f}{\partial t}$ which is the same as the continuity equation if we note that $\rho = \rho_b + \rho_f$ where ρ_b is the (unchanging) bound charge density. $\mathcal{QED}$ ✓

(*c*) Starting with Maxwell's equations in free space ($\vec{J} = 0,\ \rho = 0$), show that $\vec{E}$ and $\vec{B}$ each satisfy a wave equation. What is the speed of propagation of the resulting wave in each case? **ANSWER**: First take the time derivative of Eq. (2) to get

$$\vec{\nabla}\times\frac{\partial\vec{E}}{\partial t} = -\frac{\partial^2\vec{B}}{\partial t^2} \qquad (6)$$

(where the order of differentiation *wrt* time and space have been reversed). Then substitute $\vec{H} \equiv \vec{B}/\mu$ and $\vec{D} \equiv \epsilon\vec{E}$ into Eq. (3) and take its curl to get

$$\frac{1}{\mu}\vec{\nabla}\times\left(\vec{\nabla}\times\vec{B}\right) = \vec{\nabla}\times\vec{J}_f + \epsilon\vec{\nabla}\times\frac{\partial\vec{E}}{\partial t}\ . \qquad (7)$$

Expanding the double curl gives $\vec{\nabla}\times\left(\vec{\nabla}\times\vec{B}\right) = \vec{\nabla}\left(\vec{\nabla}\cdot\vec{B}\right) - \nabla^2 B$ and Gauss' law says $\vec{\nabla}\cdot\vec{B} = 0$, so Eq. (7) becomes

$$-\frac{1}{\mu\epsilon}\nabla^2\vec{B} = \frac{1}{\epsilon}\vec{\nabla}\times\vec{J}_f + \vec{\nabla}\times\frac{\partial\vec{E}}{\partial t}\ . \qquad (8)$$

At this point you were expected to impose the usual conditions for propagating EM waves, namely no free charges or currents ($\rho_f = \vec{J}_f = 0$). In a *conductor*, things get a bit more complicated (see Section 9.4 of the textbook). With $\vec{J}_f = 0$, Eq. (8) has the same RHS (right hand side) as the LHS of Eq. (6), yielding $\frac{\partial^2\vec{B}}{\partial t^2} = \frac{1}{\mu\epsilon}\nabla^2\vec{B}$ or

$$\boxed{\nabla^2\vec{B} - \frac{1}{v^2}\frac{\partial^2\vec{B}}{\partial t^2} = 0}$$

— *i.e.* the wave equation for $\vec{B}$, where the speed of propagation is $\boxed{v = 1/\sqrt{\mu\epsilon}}$.

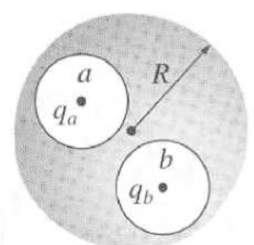
In free space, of course, $c = 1/\sqrt{\mu_0\epsilon_0} \equiv 2.99792458 \times 10^8$ m/s [exactly, by definition]. Since $\mu \geq \mu_0$ and $\epsilon \geq \epsilon_0$, $v = c/n \leq c$, where $n = \sqrt{\mu\epsilon/\mu_0\epsilon_0} \geq 1$ is the index of refraction.

For $\vec{\boldsymbol{E}}$ we take the time derivative of Eq. (3), with $\vec{\boldsymbol{H}} \equiv \vec{\boldsymbol{B}}/\mu$ and $\vec{\boldsymbol{D}} \equiv \epsilon\vec{\boldsymbol{E}}$, this time setting $\vec{\boldsymbol{J}}_f = 0$ at the outset. The result is $\vec{\boldsymbol{\nabla}} \times \frac{\partial\vec{\boldsymbol{B}}}{\partial t} = \mu\epsilon\frac{\partial^2\vec{\boldsymbol{E}}}{\partial t^2}$. Then we take the curl of Eq. (2) to get $\nabla^2\vec{\boldsymbol{E}} = \vec{\boldsymbol{\nabla}} \times \frac{\partial\vec{\boldsymbol{B}}}{\partial t}$. Again matching up RHS with LHS gives

$$\boxed{\nabla^2\vec{\boldsymbol{E}} - \frac{1}{v^2}\frac{\partial^2\vec{\boldsymbol{E}}}{\partial t^2} = 0}$$

— *i.e.* the same WAVE EQUATION for $\vec{\boldsymbol{E}}$.

2. CHARGED CONDUCTORS: Two spherical cavities, of radii a and b, are hollowed out from the interior of a solid neutral conducting sphere of radius R, as shown in the figure. There are charges q_a and q_b at the centres of the respective cavities.



(*a*) What is the electric field in the solid (shaded) conducting material?
ANSWER: $\boxed{\text{Zero}}$.

(*b*) Find the surface charges σ_a, σ_b and σ_R at the respective surfaces.
ANSWER: Since the point charges are centered on the spherical cavities, by GAUSS' LAW it takes an equal and opposite uniform surface charge to cancel $\vec{\boldsymbol{E}}$ inside the conductor. Thus $\boxed{\sigma_a = -q_a/(4\pi a^2)}$ and $\boxed{\sigma_b = -q_b/(4\pi b^2)}$. The net interior surface charge, $-(q_a + q_b)$, leaves behind a charge $+(q_a + q_b)$ which will distribute itself uniformly over the outer surface (to achieve maximum separation), giving an exterior surface charge of $\boxed{\sigma_R = (q_a + q_b)/(4\pi R^2)}$.

(*c*) What is the electric field outside the conductor at a distance $r > R$ from the centre of the large sphere?
ANSWER: $\boxed{\vec{\boldsymbol{E}} = \frac{1}{4\pi\epsilon_0}\frac{q_a + q_b}{r^2}\,\hat{\boldsymbol{r}}}$.

(*d*) What are the electric fields inside cavities a and b?
ANSWER: $\boxed{\vec{\boldsymbol{E}} = \frac{1}{4\pi\epsilon_0}\frac{q_a}{r_a^2}\,\hat{\boldsymbol{r}}_a}$ and $\boxed{\vec{\boldsymbol{E}} = \frac{1}{4\pi\epsilon_0}\frac{q_b}{r_b^2}\,\hat{\boldsymbol{r}}_b}$, where r_a and r_b are the distances from the centre of each cavity, respectively.

(*e*) What are the forces on q_a and q_b?
ANSWER: $\boxed{\text{Zero}}$. They are centred in their cavities.

(*f*) If a third charge q_c were brought near the conductor, which (if any) would change:

i. σ_a? $\boxed{\text{No}}$.

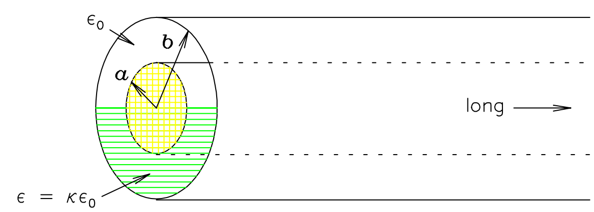
ii. σ_b? $\boxed{\text{No}}$.

iii. σ_R? $\boxed{\text{Yes}}$. An additional (nonuniform) surface charge would be required to terminate electric field lines coming from the third charge.

iv. The electric fields inside cavities a and b? $\boxed{\text{No}}$.

v. The electric field outside the conductor? $\boxed{\text{Yes}}$. Doh!

3. COAXIAL CAPACITOR: A capacitor is constructed of two very long concentric cylindrical conductors with their common axis horizontal, as shown in the diagram. The space between them is exactly half filled with a linear dielectric liquid with dielectric constant κ.



(*a*) Show that the electric field is radial and is the same in the dielectric half as in the vacuum half of the capacitor.
ANSWER: Symmetry with respect to z ensures that $\vec{\boldsymbol{E}}$ can have only radial and aximuthal components E_r and E_ϕ and that neither one changes with z. [I am using r rather than the textbook's s for the radial direction.] Meanwhile each of the *conductors* is at constant potential, forcing cylindrical symmetry upon $V(r, \phi)$ through its boundary conditions. This symmetry is passed on to $\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}V$, so there can be no aximuthal component of $\vec{\boldsymbol{E}}$: $E_\phi = 0$ and $\vec{\boldsymbol{E}}$ is radial everywhere. ✓ $\mathcal{QED}$. [Note that this does *not* mean a uniform distribution of

surface charge on the conductors! See next part.]

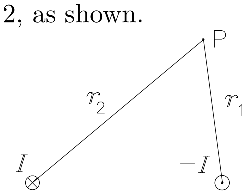
(b) Deduce the capacitance per unit length of this coaxial capacitor. **ANSWER**: A capacitor filled with a dielectric has a larger surface charge on the conductors for the same electric field in between (*i.e.* the same voltage across) so its capacitance is increased by a factor of $\kappa \equiv \epsilon/\epsilon_0$ (the dielectric constant, written "ϵ_r" in the textbook). More charge will pile up on the conductors where the dielectric is than elsewhere; σ will be uniform within each region, but a factor of κ higher where the space is filled with dielectric. The net capacitance is the *average* of what one would get for the empty capacitor and what one would get if it were completely filled:

$$\boxed{\frac{C}{L} = \frac{2\pi}{\ln(b/a)}\left[\frac{\epsilon+\epsilon_0}{2}\right] = \frac{\pi\epsilon_0(\kappa+1)}{\ln(b/a)}}$$

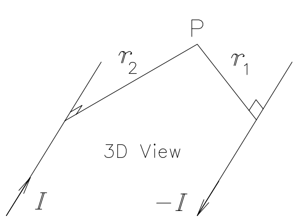
(c) If the conductors carry free charges per unit length $\pm\lambda$, find the polarization $\vec{\boldsymbol{P}}$ in the dielectric at any point a distance r from the central axis, in terms of $\epsilon_0, \kappa, \lambda$ and r. **ANSWER**: We have $\vec{\boldsymbol{P}} \equiv \vec{\boldsymbol{D}} - \epsilon_0\vec{\boldsymbol{E}} = (\kappa-1)\epsilon_0\vec{\boldsymbol{E}}$. From Gauss' law on a cylinder of length ℓ and radius r (with $a < r < b$) we have $\oint\!\!\oint \vec{(D)} \cdot d\vec{\boldsymbol{a}} = \pi r\ell(\epsilon_0 E + \epsilon E) = Q_f = \ell\lambda$. Thus $\epsilon_0 E = \frac{\lambda}{\pi r(1+\kappa)}$, giving

$$\boxed{\vec{\boldsymbol{P}} = \frac{(\kappa-1)\lambda}{\pi r(\kappa+1)}\,\hat{\boldsymbol{r}}}\,.$$

4. **LINEAR CURRENTS:** Two very long parallel wires carry equal currents $\pm I$ in opposite directions, as illustrated in the figure. Take the $\hat{\boldsymbol{z}}$ direction to be out of the page, in the direction of the current in wire 1. The field point P is located a distance r_1 from wire 1 and a distance r_2 from wire 2, as shown.



Head–on View



(a) Consider each wire separately and indicate the direction of the vector potential $\vec{\boldsymbol{A}}$ in each case. **ANSWER**: Generally

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi}\iiint \frac{\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\,d\tau'\,.$$

For thin wires we can replace the volume integral by a line integral:

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi}I\int\frac{d\vec{\boldsymbol{\ell}}'}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\,,$$

where it is usually assumed that the current loop is closed. In this case we pretend it is not, and take one wire at a time, choosing the origin at point P for simplicity ($\vec{\boldsymbol{r}} = 0$). Since $d\vec{\boldsymbol{\ell}}'$ is everywhere parallel to $\hat{\boldsymbol{z}}$, at point P $\boxed{\vec{\boldsymbol{A}}_1 = A_1\hat{\boldsymbol{z}}}$ and $\boxed{\vec{\boldsymbol{A}}_2 = -A_2\hat{\boldsymbol{z}}}$.

(b) Show that the vector potential $\vec{\boldsymbol{A}}$ at the point P is given by: $\vec{\boldsymbol{A}} = \frac{\mu_0 I}{2\pi}\ln\left(\frac{r_2}{r_1}\right)\hat{\boldsymbol{z}}$ **ANSWER**: Choose the zero of $\vec{\boldsymbol{A}}$ arbitrarily at some radius R from the wire of your choice. If we construct a rectangular loop with sides of length ℓ parallel to the wire (along $\pm\hat{\boldsymbol{z}}$) and the other sides radial from $r < R$ out to R, the magnetic flux through that loop, $\Phi_B \equiv \iint \vec{\boldsymbol{B}}\cdot d\vec{\boldsymbol{a}}$ around that loop is equal (since $\vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}}$) to $\iint(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}})\cdot d\vec{\boldsymbol{a}} = \oint\vec{\boldsymbol{A}}\cdot d\vec{\boldsymbol{\ell}}$ by Stokes' theorem. We know $\vec{\boldsymbol{B}} = \frac{\mu_0 I}{2\pi r'}\hat{\boldsymbol{\phi}}$ so $\Phi_B = \frac{\mu_0 I\ell}{2\pi}\int_r^R\frac{dr'}{r'} = \frac{\mu_0 I\ell}{2\pi}\ln\frac{R}{r}$. Meanwhile, since two sides are $\perp\vec{\boldsymbol{A}}$ and one of the others is at R where $\vec{\boldsymbol{A}} = 0$, the line integral of $\vec{\boldsymbol{A}}\cdot d\vec{\boldsymbol{\ell}}$ is just $\ell A(r)$. Setting these equal gives $\boxed{A(r) = \frac{\mu_0 I}{2\pi}\ln\frac{R}{r}}$ for the wire of your choice. The total $\vec{\boldsymbol{A}}$ is the sum of $\vec{\boldsymbol{A}}_1 + \vec{\boldsymbol{A}}_2 = [A(r_1) - A(r_2)]\hat{\boldsymbol{z}} = \frac{\mu_0 I}{2\pi}\left[\ln\frac{R}{r_1} - \ln\frac{R}{r_2}\right]\hat{\boldsymbol{z}}$ or

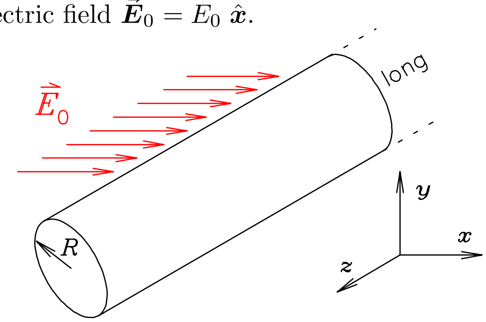
$$\boxed{\vec{\boldsymbol{A}} = \frac{\mu_0 I}{2\pi}\ln\left(\frac{r_2}{r_1}\right)\hat{\boldsymbol{z}}}\,.\ \checkmark\ \mathcal{QED}$$

(c) Show that the result in part (b) is consistent with that obtained using Ampère's law. **ANSWER**: Treat the wires separately. For the wire of your choice, in cylindrical coordinates (again using r instead of s) $\vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}} = \left[\frac{1}{r}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right]\hat{\boldsymbol{r}} + \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right]\hat{\boldsymbol{\phi}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial\phi}\right]\hat{\boldsymbol{z}}$ but $A_r = A_\phi = 0$ and $\frac{\partial A_z}{\partial\phi} = 0$, so $\vec{\boldsymbol{B}} = -\frac{\partial A_z}{\partial r}\hat{\boldsymbol{\phi}}$, or $\vec{\boldsymbol{B}} = -\frac{\mu_0 I}{2\pi}\frac{\partial}{\partial r}[\ln R - \ln r]\hat{\boldsymbol{\phi}}$, or

$$\boxed{\vec{\boldsymbol{B}} = \frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\phi}}}$$

for each wire, the same as we get from AMPÈRE'S LAW. (Note: since each wire's $\hat{\boldsymbol{\phi}}$ is different, the net field is messy looking; that's why we just did it one at a time, relying on the knowledge that we can superimpose them if needed. Also note: the derivation of $\vec{\boldsymbol{A}}$ *used* the $\vec{\boldsymbol{B}}$ above, supplied by AMPÈRE'S LAW, so consistency is hardly surprising!)

5. LAPLACE'S EQUATION: Consider an infinitely long metal pipe, of radius R, which is placed at right angles to an otherwise uniform electric field $\vec{\boldsymbol{E}}_0 = E_0\,\hat{\boldsymbol{x}}$.



Hint: Note that this situation has cylindrical symmetry (not spherical!), with no z dependence, and hence simplifies to a 2-D plane polar problem.

(*a*) What is the "uniqueness theorem" and why would you want to use it to solve for the electric potential V?
ANSWER: The UNIQUENESS THEOREM says that if you have *a* solution $V(\vec{\boldsymbol{r}})$ of Poisson's (or Laplace's) equation in some region and it has the right values at all the *boundaries* of that region, then it is *the only* solution in that region. It's obvious why this is desirable: you can *guess*, and if it works, it's right! Of course, educated guesses are a lot more efficient, and so we check our list of all *possible* solutions for the specified *symmetry* imposed by the boundary conditions. This quickly eliminates all the really dumb gueses. Then we take advantage of the linearity of the differential equation for V, which ensures that any linear combination of the above list is allowed. Usually from there we just have to find the constants that make it fit at the boundaries. What's not to like?

(*b*) What are the boundary conditions on the electric potential V? **ANSWER**: Let's measure r from the axis of the cylinder and set $\theta = 0$ in the $\hat{\boldsymbol{x}}$ direction. The surface of the conductor at $r = R$ must be an equipotential (which we are free to set equal to zero), $\boxed{V(r = R) = 0}$ independent of θ, but far from the cylinder $\vec{\boldsymbol{E}} \underset{r\to\infty}{\longrightarrow} E_0\,\hat{\boldsymbol{x}}$ so that $V(r,\theta) \underset{r\to\infty}{\longrightarrow} -E_0\,x$ or $\boxed{V(r,\theta) \underset{r\to\infty}{\longrightarrow} -E_0\,r\cos\theta}$.

(*c*) Solve Laplace's equation for the potential V outside the long metal pipe. You should obtain: $V(r,\theta) = E_0\,r\left(\frac{R^2}{r^2} - 1\right)\cos\theta$.
ANSWER: The most general form has all integer powers of $\cos\theta$ and $\sin\theta$, but the boundary condition at $r \to \infty$ tells us immediately that we need only use $n = 1$ with $\cos\theta$ and correspondingly only r or $1/r$ (or both) for the r-dependence. The most general *reasonable* solution is thus $V(r,\theta) = \left(a_1 r^1 + b_1 r^{-1}\right)\cos\theta$. Let's see if we can find values for a_1 and b_1 that satisfy the boundary conditions. As $r \to \infty$ we can ignore the effect of the second term, leaving simply $a_1 = -E_0$. Good so far. How about $V(r = R) = 0 = (-E_0 R + b_1/R)\cos\theta$? This can only be true if $b_1/R = E_0 R$ or $b_1 = E_0 R^2$. Hey, we're done!

$$\boxed{V(r,\theta) = E_0\left(\frac{R^2}{r} - r\right)\cos\theta} \,. \ \checkmark\ \mathcal{QED}$$