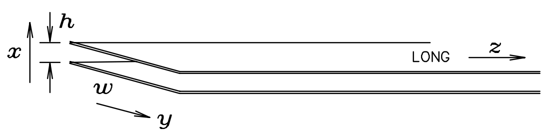
THE UNIVERSITY OF BRITISH COLUMBIA

Physics 401 Assignment # 3: Conservation Laws SOLUTIONS:

Wed. 18 Jan. 2006 — finish by Wed. 25 Jan.

1. (p. 340, Problem **7.58**) — **TRANSMISSION LINE:** A transmission line is constructed two parallel thin metal "ribbons" of width w separated by a very small distance $h \ll w$. The current travels down one strip and back along the other. In each case it spreads out uniformly over the surface of the ribbon.



(a) Find the capacitance per unit length, $\mathcal{C}$. **ANSWER**: We have a parallel-plate capacitor with a plate separation $d = h$ and an area $w\ell$, where ℓ is the length of the strip. Thus $C = \epsilon_0 A/d = \epsilon_0 w\ell/h$ and

$$\boxed{\mathcal{C} \equiv C/\ell = \epsilon_0 w/h} .$$

(b) Find the inductance per unit length, $\mathcal{L}$. **ANSWER**: Assuming the current flows to the right ($+\hat{\boldsymbol{z}}$) on the bottom strip and back ($-\hat{\boldsymbol{z}}$) on the top, AMPÈRE'S LAW gives $\vec{\boldsymbol{B}} = (\mu_0 I/w)\hat{\boldsymbol{y}}$ (uniform between the strips, zero elsewhere). For a length ℓ the resultant flux is $\Phi = Bh\ell = \mu_0(h/w)\ell I = LI$, so

$$\boxed{\mathcal{L} \equiv L/\ell = \mu_0 h/w} .$$

(c) What is the product, $\mathcal{LC}$, numerically? [1] **ANSWER**: $\mathcal{LC} = \epsilon_0(w/h)\mu_0(h/w) = \epsilon_0\mu_0$ or

$$\boxed{\mathcal{LC} = 1/c^2 = 1.11265 \times 10^{-17}\ \mathrm{s^2/m^2}} .$$

(d) If the strips are insulated from one another by a nonconducting material of permittivity ϵ and permeability μ, what is then the product $\mathcal{LC}$? What is the propagation speed? [2] **ANSWER**: We simply replace ϵ_0 by ϵ and μ_0 by μ, giving

$$\boxed{\mathcal{LC} = \epsilon\mu = 1/v^2}$$

where $v < c$ is the propagation velocity of a pulse down the line.

2. (p. 349, Problem **8.1**) — **POWER TRANSMISSION:** Calculate the power (energy per unit time) transported down the cables of Exercise 7.13 (p. 319) and Problem 7.58 (p. 340), assuming the two conductors are held at a potential difference V, and carry current I (down one and back up the other). **ANSWER**: Exercise 7.13 describes a coaxial cable with inner radius a and outer radius b. Naturally the answer should be $P = VI$ in both cases; the idea is to check this against the result calculated from $P = \iint \vec{\boldsymbol{S}} \cdot d\vec{\boldsymbol{a}}$ where $\vec{\boldsymbol{S}} = \vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}/\mu_0$ is the Poynting vector representing *energy flux* per unit time per unit area. For the coaxial cable, $\vec{\boldsymbol{E}} = \lambda\hat{\boldsymbol{r}}/2\pi\epsilon_0 r$ and $\vec{\boldsymbol{B}} = \mu_0 I\hat{\boldsymbol{\phi}}/2\pi r$ so $\vec{(S)} = \lambda I\hat{\boldsymbol{z}}/4\pi^2\epsilon_0 r^2$ and the power is $P = (\lambda I/4\pi^2\epsilon_0)\int_a^b r^{-2} \cdot 2\pi r dr$ or

$$\boxed{P = \frac{\lambda I}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right)} .$$

Is this the same as VI? We have $V = \int \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{\ell}} = (\lambda/2\pi\epsilon_0)\int_a^b r^{-1}dr = (\lambda/2\pi\epsilon_0)\ln(b/a)$ and this times I is indeed the above P. ✓ $\mathcal{QED}$

For Problem 7.58, $\vec{\boldsymbol{E}} = \sigma\hat{\boldsymbol{x}}/\epsilon_0$ and $\vec{\boldsymbol{B}} = (\mu_0 I/w)\hat{\boldsymbol{y}}$ are uniform and mutually perpendicular, making $\vec{\boldsymbol{S}} = \vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}/\mu_0 = (\sigma I/\epsilon_0 w)\hat{\boldsymbol{z}}$ and thus $P = \iint \vec{\boldsymbol{S}} \cdot d\vec{\boldsymbol{a}} = (\sigma I/\epsilon_0 w)(wh)$ or

$$\boxed{P = \frac{\sigma I h}{\epsilon_0}} .$$

Compare $V = Eh = \sigma h/\epsilon_0$ so that $VI = \sigma I h/\epsilon_0$. ✓ $\mathcal{QED}$

3. (p. 357, Problem **8.5**) — **FORCE on a PARALLEL PLATE CAPACITOR:** Consider a semi-infinite parallel plate capacitor (far from the edges), with the lower plate (at $z = -d/2$) carrying a uniform charge density $-\sigma$ and the upper plate (at $z = +d/2$) carrying a uniform charge density $+\sigma$.

(a) Determine all nine elements of the stress tensor in the region between the plates.

[1] $\mathcal{L}$ and $\mathcal{C}$ will, of course, vary from one kind of transmission line to another, but their *product* is a universal constant — check, for example, the cable in Exercise 7.13 on p. 319 — provided the space between the conductors is a vacuum. In the theory of transmission lines, this product is related to the speed at which a pulse propagates down the line ($v = 1/\sqrt{\mathcal{LC}}$).

[2] *Hint:* see Exercise 4.6 on p. 183; by what factor does $\mathcal{L}$ change when an inductor is immersed in linear material of permeability μ?

Display your answer as a 3×3 matrix:

$$\begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$$

ANSWER: Generally

$$T_{ij} = \epsilon_0(E_iE_j - \delta_{ij}E^2/2) + (B_iB_j - \delta_{ij}B^2/2)/\mu_0 \; .$$

In this case $\vec{\boldsymbol{B}} = 0$ and $\vec{\boldsymbol{E}} = -(\sigma/\epsilon_0)\hat{\boldsymbol{z}}$ has only one component (E_z), so all off-diagonal terms are zero and

$$\boxed{T_{ij} = \frac{\sigma^2}{2\epsilon_0}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{pmatrix}} \; .$$

(*b*) Use Eq. (8.22) on p. 353 to determine the force per unit area on the top plate. Compare Eq. (2.51) on p. 103.
ANSWER: Since $\vec{\boldsymbol{B}} = 0$, $\vec{\boldsymbol{S}} = 0$, leaving $F_j = \oiint T_{ij}\,da_i$. A surface enclosing *both* plates will yield a zero result, since there is no field outside the capacitor. What we want is a surface enclosing just the *top* plate. Its shape above the plate where $\vec{\boldsymbol{E}} = 0$ doesn't matter, but inside the gap it should be normal to $\vec{\boldsymbol{E}}$ and parallel to the plate — *i.e.* $d\vec{\boldsymbol{a}} = -\hat{\boldsymbol{z}}da$ (remember, $d\vec{\boldsymbol{a}}$ always points *out* of the enclosed region). Thus Eq. (8.22) is reduced to $F_z = -AT_{zz} = -\sigma^2 A/2\epsilon_0$ and

$$\boxed{\vec{\boldsymbol{\mathcal{F}}} \equiv \frac{\vec{\boldsymbol{F}}}{A} = -\frac{\sigma^2}{2\epsilon_0}\hat{\boldsymbol{z}}} \; ,$$

in agreement with Eq. (2.51). ✓ $\mathcal{QED}$

(*c*) What is the momentum per unit area, per unit time, crossing the xy plane (or any other plane parallel to that one, between the plates)? **ANSWER**: Really this is just a question of momentum conservation. The upper plate feels a downward force (and the lower plate an equal and opposite upward one); this force is **transmitted** by the electric field: you may think of the lower plate "emitting" an electromagnetic field with momentum flux $\frac{\partial\vec{\boldsymbol{\mathcal{P}}}}{\partial t}$ per unit area and time, and the upper plate "absorbing" same:

$$\boxed{\frac{\partial\vec{\boldsymbol{\mathcal{P}}}}{\partial t} = \vec{\boldsymbol{\mathcal{F}}} = -\frac{\sigma^2}{2\epsilon_0}\hat{\boldsymbol{z}}} \; .$$

(*d*) At the plates this momentum is absorbed, and the plates recoil (unless there is some other force holding them in position). Find the recoil force per unit area on the top plate, and compare your answer to that in part (*b*). [3] **ANSWER**: It's the <u>**same thing**</u>, we already said that. There are two conceptual challenges to this picture: First, we are not used to "things" whose momentum is *toward* their emitter and *away from* their absorber. Switching the roles of the two plates is no help; the same conundrum persists. Second, we have already noted that $\vec{\boldsymbol{S}} = 0$ (no magnetic field). So the momentum is *not* being transmitted as a Poynting vector. It is in the stress tensor itself (see p. 356). (By asking the same question three times in different guises, Griffiths is trying to force you to reconcile these notions in your own mind. I hope it worked. :-)

4. (p. 361, Problem **8.9**) — **SOLENOID and RING:** A very long solenoid of radius a, with n turns per unit length, carries a current I_S. Coaxial with the solenoid, at radius $b \gg a$, is a circular ring of wire with resistance R. When the current in the solenoid is gradually decreased, a current I_r is induced in the ring.

(*a*) Calculate I_r in terms of dI_S/dt.
ANSWER: We have within the solenoid a uniform magnetic field $\vec{\boldsymbol{B}} = \mu_0 n I_S \hat{\boldsymbol{z}}$, giving a flux $\Phi = \pi a^2 B = \mu_0 n\pi a^2 I_S = LI_S$ in the positive z direction. If I_S decreases, a current $I_r = \mathcal{E}/R$ will flow in such a direction as to replace the missing flux — *i.e.* in the same sense as the original current in the solenoid. Here $\mathcal{E} = -\dot{\Phi} = -\mu_0 n\pi a^2 \partial I_S/\partial t$, so that

$$\boxed{I_r = -\frac{\mu_0 n\pi a^2}{R}\,\frac{\partial I_S}{\partial t}} \; .$$

(*b*) The power ($I_r^2 R$) delivered to the ring must have come from the solenoid. Confirm this by calculating the Poynting vector just outside the solenoid, where the *electric* field is due to the changing flux in the solenoid and the *magnetic* field is due to the current in the ring. Integrate over the entire surface of the solenoid, and check that you recover the correct total power. **ANSWER**: In the above calculation (which required only first year Physics methods) we were careful not to mix up the "cause" (the changing magnetic field of the solenoid) with the "effect" (the magnetic field generated by I_r in the ring).

[3] *Note:* this is not an *additional* force, but rather an alternative way of calculating the *same* force — in (*b*) we got it from the force law, and in (*d*) we got it from conservation of momentum.

But when we think in terms of "the electromagnetic field" in calculating the Poynting vector (or, for that matter, the stress tensor) there is no such separation: we must use the *total* field(s) at any given time. Outside the solenoid there is no $\vec{B}$ from the solenoid itself, but the current in the ring generates a field $\vec{B} = \frac{\mu_0 I_r}{2} \frac{b^2}{(b^2+z^2)^{3/2}} \hat{z}$ (see Example 5.6 on p. 218). Meanwhile $\mathcal{E} = \oint \vec{E} \cdot d\vec{\ell} = -\mu_0 n \pi a^2 \partial I_S/\partial t$, and (by symmetry) $\vec{E} = -E(r)\hat{\phi}$, so around any loop at r we have $\oint \vec{E} \cdot d\vec{\ell} = -2\pi r E(r)$ $= -\mu_0 n \pi a^2 \partial I_S/\partial t$, giving $\vec{E} = -\frac{\mu_0 n a}{2} \frac{\partial I_S}{\partial t} \hat{\phi}$. Putting these together gives $\vec{S} = \vec{E} \times \vec{B}/\mu_0$ $= -\frac{1}{\mu_0} \frac{\mu_0 n a}{2} \frac{\partial I_S}{\partial t} \frac{\mu_0 I_r b^2}{2\,(b^2+z^2)^{3/2}} \hat{r}$. Using the earlier result we can substitute $-(I_r R/\mu_0 n \pi a^2)$ for $\partial I_S/\partial t$ to get $\vec{S} = I_r^2 R \frac{b^2}{4\pi a\,(b^2+z^2)^{3/2}} \hat{r}$. Now we integrate $\vec{S} \cdot d\vec{a} = S da$ over the surface of the solenoid to get the net power P "sent" to the ring:

$$P = I_r^2 R \frac{b^2}{4\pi a} \int_{-\infty}^{+\infty} \frac{2\pi a\, dz}{(b^2+z^2)^{3/2}}$$

$= I_r^2 R \frac{b^2}{2} \int_{-\infty}^{+\infty} \frac{dz}{(b^2+z^2)^{3/2}}$. Looking up the integral (`http://integrals.wolfram.com/index.jsp` is a big help!) we have

$$\int_{-\infty}^{+\infty} \frac{dz}{(b^2+z^2)^{3/2}} = \left[\frac{z}{b^2\,(b^2+z^2)^{1/2}} \right]_{-\infty}^{+\infty} = \frac{2}{b^2},$$

giving

$$\boxed{P = I_r^2 R}\ . \checkmark$$

(This sure is doing it the hard way, but it's nice to know that $\vec{S}$ really is transmitting power.)

5. PHOTON DRIVE: Rocket ships propelled by photon drives often appear in science fiction novels and movies. The idea is to generate thrust by expelling photons. Since the "exhaust velocity" of photons is as high as you can get (the speed of light), you might expect photon drive rockets to outperform conventional rockets.

(*a*) Calculate the *power* you'd need to produce 1 Newton of thrust with a photon drive rocket. How does this compare with the typical output of BC's huge Stave Lake power station, which has a peak capacity of about 200 MW? **ANSWER**: We did something like this in class: if $\vec{S}$ is power per unit area and $\vec{S}/c^2$ is momentum density per unit volume, then $\vec{S}/c$ is the "radiation pressure" (force per unit area) and the relationship between net force F and net power P is just $F = P/c$. So for a 1 N thrust you'd need about 3×10^8 W or $\boxed{\text{300 MW}}$. Stave Lake could manage 2/3 N.

(*b*) What accelerations would result if your power source provided 200 MW and the total mass of the rocket were 20,000 kg? **ANSWER**: From $a = F/m = 2/(3 \times 2 \times 10^4)$ we get $\boxed{a = 0.333 \times 10^{-4}\ \text{m/s}^2}$. Pretty puny.

(*c*) In spite of these numbers, the photon drive offers one very attractive advantage over conventional rockets, especially for *long* space voyages. What is it? **ANSWER**: It keeps going and going and going and .... If it weren't for relativity, we would reach c after only 285,000 years! Hmm, maybe if we use antimatter annihilation....