THE UNIVERSITY OF BRITISH COLUMBIA

Physics 401 Assignment # 12: RADIATION 2

SOLUTIONS:

Wed. 29 Mar. 2006 — finish by Wed. 5 Apr.

1. (p. 450, Problem **11.3**) — **Radiation Resistance of a Cell Phone:** Find the **radiation resistance** of the wire joining the two ends of the oscillating electric dipole described in Section 11.1.2. (This is the resistance that would give the same average power loss — to heat — as the oscillating dipole in *fact* puts out in the form of radiation.) Show that $R = 790(d/\lambda)^2\ \Omega$, where λ is the wavelength of the radiation. For the wires in an ordinary cell phone (say, $d = 5$ cm), should you worry about the radiation contribution to the total resistance? Does it matter whether your cell phone uses the 900 MHz band or the 1.9 GHz band?[1]

ANSWER: For a simple resistor R driven by a power supply that moves charge back and forth in an oscillation $q(t) = q_0 \cos(\omega t)$, we have a current $I(t) = -\omega q_0 \sin(\omega t)$ and a power $P(t) = I^2(t)R = \omega^2 q_0^2 \sin^2(\omega t)R$ which averages to $\langle P\rangle = \frac{1}{2}\omega^2 q_0^2 R$. Thus we can associate $R_{\rm eff} = 2\langle P\rangle/\omega^2 q_0^2$. Plugging in the average power radiated by the electric dipole, $\langle P\rangle = \dfrac{\mu_0\, p_0^2\, \omega^4}{12\pi c}$, with $p_0 = q_0 d$, we get $R_{\rm eff} = \dfrac{2\langle P\rangle}{\omega^2 q_0^2} = \dfrac{\mu_0\, q_0^2\, d^2\, \omega^4}{6\pi c \omega^2 q_0^2} = \dfrac{\mu_0\, d^2\, \omega^2}{12\pi c}$. Since $\lambda = 2\pi c/\omega$, we can substitute $\omega = 2\pi c/\lambda$ to get $R_{\rm eff} = \dfrac{\mu_0\, d^2\, 4\pi^2 c^2}{6\pi c \lambda^2} = \dfrac{2}{3}\mu_0\, \pi\, c \left(\dfrac{d}{\lambda}\right)^2$. The coefficient $2\mu_0\pi c/3 = 789$ N-A^{-2}m-s^{-1}, whose units are equivalent to W/A^{-2} or Ω, leaving

$$\boxed{R_{\rm eff} = [789\ \Omega] \times \left(\frac{d}{\lambda}\right)^2}.$$

For 900 or 1900 MHz, we have $\lambda = c/\nu = 33.3$ cm or 15.8 cm, respectively,[2] giving *radiation resistances* of $\boxed{17.8\ \Omega}$ or $\boxed{79.2\ \Omega}$, respectively. Neither is a huge resistance, but both are certainly larger than that of the *wires* in your cell phone. The power is thus used quite efficiently. (Very little goes into useless heat; almost all is transmitted!) This is even more true of the higher frequency band: whatever transmission intensity is required, it can be realized with a smaller I.

2. (p. 454, Problem **11.6**) — **Radiation Resistance of a Magnetic Dipole Antenna:** Find the radiation resistance for the oscillating magnetic dipole shown in Fig. 11.8. Express your answer in terms of λ and b, and compare the radiation resistance of the *electric* dipole.[3]

ANSWER: For the magnetic dipole, $\langle P\rangle = \dfrac{\mu_0\, m_0^2\, \omega^4}{12\pi c^3}$ where $mu_0 = \pi b^2 I_0$. Again setting this equal to $\frac{1}{2}I_0^2 R_{\rm eff}$, we get $R_{\rm eff} = 2\langle P\rangle/I0^2 = \dfrac{2\mu_0(\pi b^2 I_0)^2\omega^4}{12\pi c^3 I_0^2} = \dfrac{\mu_0 \pi b^4 \omega^4}{6c^3}$. Again substituting $2\pi c/\lambda$ for ω, we get $R_{\rm eff} = \dfrac{8}{3}\mu_0\pi^5 c\left(\dfrac{b}{\lambda}\right)^4$. The coefficient $\frac{8}{3}\mu_0\pi^5 c = 3.074 \times 10^5\ \Omega$, so

$$\boxed{R_{\rm eff} = [3.074 \times 10^5\ \Omega] \times \left(\frac{b}{\lambda}\right)^4}.$$

In this case, for a given frequency, the radiation resistance increases as the *square of the area of the loop*. For $\lambda = 33.3$ cm or 15.8 cm, a 2.5 cm radius loop would have $R_{\rm eff} = 9.77\ \Omega$ or $192.7\ \Omega$, respectively. Thus the magnetic dipole antenna is similar to the electric dipole antenna at this size and frequency, but is much more strongly size- and frequency-dependent.

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[1] You might also want to calculate the intensity of your cell phone's transmission signal at a distance of 10 cm (*i.e.* in your brain while you hold it to your ear). This is a topic upon which a great deal has been written. Just Google it! But it's not part of this assignment.

[2] Note that this scenario barely satisfies the "**slow** approximation" $d \ll \lambda$ used to derive the formula for $\langle P\rangle >$ for the radiating electric dipole. For a tuned half-wave antenna ($d = \lambda/2$) the approximation is completely invalid.

[3] You should get $R = 3 \times 10^5 (b/\lambda)^4\ \Omega$.

3. (p. 464, Problem **11.13**) — **Nonrelativistic Bremsstrahlung Radiation:**

(*a*) Suppose an electron decelerates at a constant rate a from some initial velocity v_0 down to zero. What fraction of its initial kinetic energy is lost to EM radiation? (The rest is absorbed by whatever mechanism keeps the acceleration constant.) Assume $v_0 \ll c$ (nonrelativistic case) so that the Larmor formula can be used.[4]

ANSWER: The LARMOR FORMULA says $P = \mu_0 q^2 a^2 / 6\pi c$. This is expended for a time $t = v_0/a$, giving a total radiated energy $E = Pt = \mu_0 q^2 v_0 a / 6\pi c$. The initial kinetic energy $K_0 = \frac{1}{2} m v_0^2$. Thus the fraction lost to EM radiation is $f_{\rm rad} = E/K_0 = 2\mu_0 q^2 v_0 a / 6\pi c m v_0^2$ or

$$\boxed{f_{\rm rad} = \frac{\mu_0 q^2 a}{3\pi m c v_0}}.$$

(*b*) To get a sense of the numbers involved, suppose the initial velocity is thermal[5] (around 10^5 m/s) and the distance over which the electron decelerates to rest is 30 Å. What can you conclude about radiation losses for electrons in an ordinary conductor?

ANSWER: Using $v_0^2 = 2ad$ with $v_0 = 10^5$ m/s and $d = 3 \times 10^{-9}$ m, we have $a = v_0^2/2d = 10^{10}/6 \times 10^{-9} = 1.67 \times 10^{18}$ m/s^2. With $q = -1.602 \times 10^{-19}$ C and $m = 0.911 \times 10^{-30}$ kg, we get $\boxed{f_{\rm rad} = 2.09 \times 10^{-10}}$. (Not much!) The true picture is much stranger, of course; electrons are not localized point charges following classical trajectories, they are described by extended wavefunctions and do not radiate at all in things like atoms (luckily!).

[4]Relativistic electrons radiate furiously; this is known as *Bremsstrahlung* (German for "braking radiation", doh!) and is an important mechanism for energy loss of high energy electrons.

[5]This thermal velocity corresponds to about 330 K, not far above room temperature, and so appears realistic. In point of fact, the conduction electrons in a good metal have velocities on the order of $10^{-3}c$, thanks to the Pauli exclusion principle. However, their quantum mechanical wavefunctions are extended over distances large compared to 30 Å, and this classical picture of an accelerated point charge has to be reformulated with a quantum version. The present approximation is a reasonable compromise.

4. **Half-Wave Antenna:** Consider a half-wave linear antenna of length ℓ, with current $I(z,t) = I_0 \cos kz \sin \omega t$, where $k = \pi/\ell$.

(*a*) Show that the linear charge density is $\lambda(z,t) = (I_0/c) \sin kz \cos \omega t$, (*i.e.* the charge density is maximum at the times when the current is zero.)

ANSWER: For a half-wave antenna $\ell = \lambda/2$ and $k = \pi/\ell$. Charge conservation requires $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{J}} + \dfrac{\partial \rho}{\partial t} = 0$. For a 1-dimensional wire with a current $I(z,t)$ flowing in the $\hat{\boldsymbol{z}}$ direction, the same logic demands $dI/dz + d\lambda/dt = 0$ where $\lambda \equiv |dq/dz|$. Thus $d\lambda = -k I_0 \sin kz \sin \omega t \; dt$. Integrating,

$$\lambda = -\frac{k}{\omega} I_0 \sin kz \int \sin u \; du$$

where $u \equiv \omega t$ and $\sin u \; du = -d \cos u$, giving $\boxed{\lambda(z,t) = \dfrac{I_0}{c} \sin kz \cos \omega t}$. ✓ Note that λ is also maximum at the *places* where the current is zero.

(*b*) If an FM radio station broadcasts at a frequency of 10 MHz with a power of 10 kW from a half-wave antenna, how long must the antenna be? What is the current?

ANSWER: The length is simple: $\ell = \lambda/2$ where $\lambda = c/\nu = 3 \times 10^8/10^7 = 30$ m. Thus $\boxed{\ell = 15 \text{ m}}$. The current we can get from the power, using $\langle P \rangle \approx 1.22 \, \dfrac{\mu_0 I_0^2 c}{4\pi}$ $= 10^4$ W. Thus $I_0^2 = \dfrac{4\pi \times 10^4}{1.22 \mu_0 c}$ or $I_0 = \sqrt{\dfrac{1.257 \times 10^5}{460}}$ or $\boxed{I_0 = 16.54 \text{ A}}$.