Lecture # 2

- **Logistics & Leftovers:** ***WebCT*** **?**
- **Review of Electrostatics & Magnetostatics, cont'd**
  - **Coulomb's (↔ Gauss') Law & Biot-Savart (↔ Ampère's) Law**
  - "TOOLKIT" for **Electrostatics**:
    - Electrostatics and **Conductors**
    - Equations of **Poisson** & **Laplace**
    - Method of **Images**
    - **Multipole** expansions
  - **... for Magnetostatics:**
    - The Vector Potential ***A***
    - **Multipole** expansions
    - . . .

Conductors & Electrostatics

$\mathbf{J} = \sigma \mathbf{E}$ until charge redistributes itself to cancel out $\mathbf{E}$. Consequently

$\mathbf{E} = 0$ inside a conductor. Since $\mathbf{E} = -\vec{\nabla} V$, $V = \text{const.}$ in a conductor.

By Gauss' law, if $\mathbf{E} = 0$ then $\rho = 0$ as well. All charges go to <u>surfaces</u>.

At <u>any interface</u>, $\mathbf{E}_{above} - \mathbf{E}_{below} = \sigma$ (σ = surface charge density)‡.

Just <u>outside a conductor</u>, $E_{\|} = 0$ & $E_{\perp} = \sigma$.

Thus the surfaces of conductors are key **boundaries** between regions where (usually) Laplace's equation $\nabla^2 V = 0$ applies. See next page.

Note: all the above assumes **electrostatics** (no steady currents applied)!

‡ Notational ambiguity: σ the surface charge density vs. σ the conductivity!

Solutions to Laplace's Equation: $\nabla^2 V = 0$

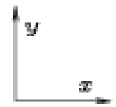
In general, $\nabla^2 V = \frac{\rho}{\epsilon_0}$

(Poisson's equation)

but most practical problems involve free space, dielectrics or conductors, where $\rho = 0$ in the regions of interest.

We solve the **differential equation** by **separation of variables** in an appropriate coordinate system, then **try** a linear combination of all the (finite variety of) solutions in that geometry, using the **equipotentials** of conducting surfaces as **boundary conditions**.

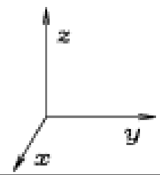
2D Cartesian:



$$\nabla^2 V \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$$V(x,y) = \left.\begin{matrix} x \\ 1 \end{matrix}\right\} \left.\begin{matrix} y \\ 1 \end{matrix}\right\} + \left.\begin{matrix} e^{kx} \\ e^{-kx} \end{matrix}\right\} \left.\begin{matrix} \cos ky \\ \sin ky \end{matrix}\right\} \quad + \text{permutations } (x \leftrightarrow y).$$

3D Cartesian:

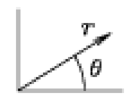


$$\nabla^2 V \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$V(x,y,z) = \left.\begin{matrix} x \\ 1 \end{matrix}\right\} \left.\begin{matrix} y \\ 1 \end{matrix}\right\} \left.\begin{matrix} z \\ 1 \end{matrix}\right\} + \left.\begin{matrix} x \\ 1 \end{matrix}\right\} \left.\begin{matrix} \cos py \\ \sin py \end{matrix}\right\} \left.\begin{matrix} e^{qz} \\ e^{-qz} \end{matrix}\right\} + \left.\begin{matrix} e^{px} \\ e^{-px} \end{matrix}\right\} \left.\begin{matrix} \cos qy \\ \sin qy \end{matrix}\right\} \left.\begin{matrix} \cos \sqrt{p^2 - q^2}\, z \\ \sin \sqrt{p^2 - q^2}\, z \end{matrix}\right\}$$

$$+ \text{ all permutations } \{x, y, z\}.$$

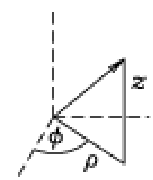
2D Plane Polar:



$$\nabla^2 V \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0$$

$$V(r,\theta) = \left.\begin{matrix} \ln r \\ 1 \end{matrix}\right\} + \left.\begin{matrix} r^n \\ r^{-n} \end{matrix}\right\} \left.\begin{matrix} \cos n\theta \\ \sin n\theta \end{matrix}\right\}$$

3D Cylindrical:

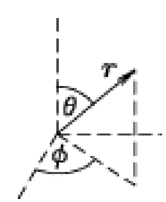


$$\nabla^2 V \equiv \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$V(\rho,\phi,z) = \left.\begin{matrix} J_n(k\rho) \\ N_n(k\rho) \end{matrix}\right\} \left.\begin{matrix} \cos n\phi \\ \sin n\phi \end{matrix}\right\} \left.\begin{matrix} e^{kz} \\ e^{-kz} \end{matrix}\right\}$$

where $J_n(k\rho) \rightarrow$ Bessel functions and $N_n(k\rho) \rightarrow$ Neumann functions.

3D Spherical:



$$\nabla^2 V \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

$$V(r,\theta,\phi) = \left.\begin{matrix} r^l \\ r^{-(l+1)} \end{matrix}\right\} \left.\begin{matrix} P_l^m(\cos\theta) \\ Q_l^m(\cos\theta) \end{matrix}\right\} \left.\begin{matrix} \cos m\phi \\ \sin m\phi \end{matrix}\right\}$$

where $P_l^m(\cos\theta)$ are associated Legendre polynomials

and $Q_l^m(\cos\theta)$ are associated Legendre polynomials of the second kind.

If axial symmetry then $V(r,\theta,\phi) = \left.\begin{matrix} r^l \\ r^{-(l+1)} \end{matrix}\right\} \left.\begin{matrix} P_l(\cos\theta) \\ Q_l(\cos\theta) \end{matrix}\right\}$

where $P_l(\cos\theta)$ are Legendre polynomials and $Q_l(\cos\theta)$ are Legendre polynomials of the second kind.

Method of Images

If a combination of the field due to real charges and that due to "pretend charges" **inside** the conductor would give a new field that is everywhere **perpendicular to the surface of the conductor** (making that surface an **equipotential**), then that is the field in the region **outside** the conductor. (Uniqueness theorem)

Not a useful "brute force" technique, but great for "**tricks**"!

Examples:

Point charge beside a conducting **plane**

Point charge beside a conducting **sphere**

The Vector Potential

Just as we set $\mathbf{E} = -\vec{\nabla} V$, we can express $\mathbf{B}$ as the **curl** of a **vector** potential $\mathbf{A}$: $\mathbf{B} = \vec{\nabla} \times \mathbf{A}$. Plugging this into Ampère's law yields $\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} + \vec{\nabla} \times (\vec{\nabla} \cdot \mathbf{A})$. We can always choose $\vec{\nabla} \cdot \mathbf{A} = 0$ to make that last term go away, leaving $\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$, which (by analogy with Poisson's equation for V) has the general solution

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi} \iiint \frac{\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}}\,')}{\mathscr{r}}\, d\tau'$$

Multipole Expansions

When the test point at $\boldsymbol{r}$ is far away and the source region (the range of $\boldsymbol{r'}$) is tiny by comparison $(\boldsymbol{r'} \ll \boldsymbol{r})$ we can treat the source region as "the origin" and expand the potentials in powers of $(\boldsymbol{r'}/\boldsymbol{r})$, yielding for the scalar (electrostatic) potential

$$V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(\vec{\boldsymbol{r}}')}{\mathscr{r}}\, d\tau' \;=\; \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{(n+1)}} \iiint (r')^n\, P_n(\cos\theta')\, \rho(\vec{\boldsymbol{r}}')\, d\tau'$$

and for the vector (magnetostatic) potential

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi} \iiint \frac{\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}}')}{\mathscr{r}}\, d\tau' \;=\; \frac{\mu_0}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{(n+1)}} \iiint (r')^n\, P_n(\cos\theta')\, \vec{\boldsymbol{J}}(\vec{\boldsymbol{r}}')\, d\tau'$$

where $P_n(\cos\theta')$ are Legendre polynomials.