"Laws" of Algebra

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In Algebra we learn to "solve" equations. What does that mean? Usually it means that we are to take a (relatively) complicated equation that has the "unknown" (often but not always called "x") scattered all over the place and turn it into a (relatively) simple equation with x on the left-hand side by itself and a bunch of other symbols (*not* including x) on the right-hand side of the "=" sign. Obviously this particular *format* is "just" a convention. But the *idea* is independent of the representation: we want to "solve" for the "unknown" quantity, in this case x, in terms of whatever else is in the equation: *numbers* like $1, 2, 3 \ldots$ or named *constants* like $a, b, c, \ldots$.

1 Operations and Notation

Most algebra involves only a few simple operations:

- **Equality:** If we write $a = b$ we are saying that a and b are the same *kind* of thing and are *exactly the same size*.
- **Equivalence:** If we write $a \equiv b$, we are saying that a and b are *the same thing*. This may sound like the same thing as *equality*, but it's actually much stronger. Below it will be applied to equivalent *notations*.[1]
- **Addition:** If a and b are entities of the same type (usually just numbers), we can **add** them together as $a + b$ to get a new number or entity of the same type, called their **sum**. Example: $1 + 2 = 3$.
- **Subtraction:** By the same token, we can **subtract** b from a to get their **difference**, $a - b$. Example: $3 - 1 = 2$.
- **Multiplication:** The **product** of a and b is written as either $a \times b$ or $a \cdot b$ or just ab, with the understanding that each entity is represented by a single character. Example: $2 \times 3 = 6$.
- **Division:** Just as *subtraction* is sort of the opposite of *addition*, *division* (written $\frac{a}{b}$ or a/b or $a \div b$) is sort of the opposite of *multiplication*. Example: $\frac{6}{2} \equiv 6/2 \equiv 6 \div 2 = 3$.
- **Powers:** We can multiply a by *itself* n times (also called "raising a to the power n) to get a^n. Example: $2^5 = 16$.
- **Roots:** In something like the opposite of raising a to the power n, we can find the n^{th} **root** of b, written $\sqrt[n]{b} \equiv b^{1/n}$. Example: $16^{1/5} \equiv \sqrt[5]{16} = 2$.

[1] The above operations are described using formal Mathematical notation and symbols that are easy to write longhand or typeset using LaTeX, but are notoriously difficult to express neatly in `HTML` or in the restricted `ASCII` character set used by computers. Most computer programming languages have `ASCII` conventions equivalent to the Mathematical symbols and operations. For instance, *equivalence* ($a \equiv b$) is expressed `A == B` in most programming languages; *multiplication* ($a \times b$) is written `A*B`; *division* is simply `A/B`; *powers* are almost universally expressed as `A**N` and *roots* as `B**(1/N)`. The Algebra tutorial is (of necessity) expressed in these forms.

2 LAWS

There are a few basic rules we use to "solve" problems in Algebra; these are called "laws" by Mathematicians who want to emphasize that you are not to question their content or representation.

- **Definition of Zero:**

$$a - a = 0 \tag{1}$$

- **Negative Values:** Along with the definition of zero, the *subtraction* operation allows us to assign a *negative value* to the expression $-a$ taken as a separate entity. That means we can think of $a - a$ as $a + (-a)$; *i.e.* if we *add* $-a$ to a we get zero again. Note that $-(-a) = a$: in mathematics, a *double negative* really **is** positive. Thus if a *itself* has a negative value, $-a$ is *positive*, and $a - a = 0$ still holds.[2]

- **Definition of Unity:**

$$\frac{a}{a} = 1 \tag{2}$$

- **Commutative Laws:**[3]

$$a + b = b + a \tag{3}$$

and

$$ab = ba \tag{4}$$

- **Distributive Law:**

$$a(b + c) = ab + ac \tag{5}$$

- **Sum or Difference of Two Equations:** Adding (or subtracting) the same thing from both sides of an equation gives a new equation that is still OK.

$$\begin{array}{rrcl} & x \quad -a = & b & \\ +\,(& a = & a &) \\ & x \qquad = & b + a & \end{array} \tag{6}$$

$$\begin{array}{rrcl} & x \quad +c = & d & \\ -\,(& c = & c &) \\ & x \qquad = & d - c & \end{array} \tag{7}$$

- **Product or Ratio of Two Equations:** Multiplying (or dividing) both sides of an equation by the same thing also gives a new equation that is still OK.

$$\begin{array}{rrcl} & x/a = & b & \\ \times\,(& a = & a &) \\ & x = & ab & \end{array} \tag{8}$$

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[2] Does this seem a bit circular? Right you are! It is!

[3] Note that *division* is *not* commutative: $a/b \neq b/a$! Neither is *subtraction*, for that matter: $a - b \neq b - a$. The Commutative Law for *multiplication*, $ab = ba$, holds for ordinary numbers (real, imaginary and complex) but it does *not* necessarily hold for all the mathematical "things" for which some form of "multiplication" is defined! For instance, the *group* of *rotation operators* in 3-dimensional space is *not* commutative — think about making two successive rotations of a rigid object about perpendicular axes in different order and you will see that the final result is different! This seemingly obscure property turns out to have fundamental significance.

$$
\begin{array}{rrcl}
 & cx = & d & \\
\div\, (& c = & c &) \\
 & x = & d/c &
\end{array}
\tag{9}
$$

- **Imaginary and Complex Numbers:** So far we have limited ourselves to the *real numbers*. In that domain, $\sqrt{-1}$ is undefined: there is no real number that will yield -1 when squared. One imagines a particularly persistent student insisting, "But what if there *were* such a number?" The teacher would grumble, "You certainly have an active imagination!" And the student would say, 'Fine. Let's call it an *imaginary* number, and call it "i" for short!' The inclusion of multiples of i more than doubles the domain of algebra, since it means we can also have *combinations* of real and imaginary numbers, $z = a + ib$. These are called *complex* numbers.

These "laws" may seem pretty trivial (especially the first two) but they define the rules of Algebra whereby we learn to manipulate the form of equations and "solve" Algebra "problems." We quickly learn equivalent *shortcuts* like "moving a factor from the bottom of the left-hand-side [often abbreviated LHS] to the top of the right-hand side [RHS]" or "moving a constant from the LHS to the RHS and changing its sign:"

$$
\frac{x-a}{b} = c + d \quad \Rightarrow \quad x - a = b(c+d) \quad \Rightarrow \quad x = a + b(c+d) \tag{10}
$$

and so on; but each of these is just a well-justified concatenation of several of the fundamental steps.

You may ask, "Why go to so much trouble to express the obvious in such formal terms?" Well, as usual, the obvious is not necessarily the truth. While the real, imaginary and complex numbers may all obey these simple rules, there are perfectly legitimate and useful fields of "things" (usually some sort of *operators*) that do *not* obey all these rules, as we may see later. It is generally a good idea to be aware of your own assumptions; we haven't the time to keep reexamining them constantly, so we try to state them as plainly as we can and keep them around for reference "just in case...."

3 The Quadratic Theorem

"I'm thinking of a number, and its name is 'x' ..." So if

$$
ax^2 + bx + c = 0, \tag{11}
$$

what is x? Well, we can only say, "It depends." Namely, it depends on the values of a, b and c, whatever they are. Let's suppose the *dimensions* of all these "parameters" are mutually consistent[4] so that the equation makes sense. Then "it can be shown" (a classic phrase if there ever was one!) that the "answer" is *generally*[5]

$$
\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}} \tag{12}
$$

[4]In Mathematics we never worry about such things; all our symbols represent *pure numbers*; but in Physics we *usually* have to express the value of some physical quantity in units which make sense and are consistent with the units of other physical quantities symbolized in the same equation!

[5]The $\pm$ symbol means that *both* signs ($+$ and $-$) should represent legitimate answers.

This formula (and the preceding equation that defines what we mean by a, b and c) is known as the QUADRATIC THEOREM, so called because it offers "the answer" to *any* quadratic equation (*i.e.* one containing powers of x up to and including x^2). The power of such a *general* solution is prodigious. (Work out a few examples!)

This also suggests an interesting new way of looking at the relationship between x and the *parameters* a, b and c that determine its value(s). Having x all by itself on one side of the equation and no x's anywhere on the other side is what we call a "solution" in Algebra. Let's make a compact version of this sort of equation:

"I'm thinking of a number, and its name is 'y' ..." So if $y = f(x)$, what is y? The answer is again, "It depends!" [In this case, upon the value of x and the detailed form of the *function* $f(x)$] ... and that leads us on into a new subject: CALCULUS!