Easy Calculus

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In a *stylistic* sense, Algebra starts to become Calculus when we write the preceding example, $y = x^2$, in the form

$$y(x) = x^2$$

which we read as "y of x equals x squared." This is how we signal that we mean to think of y as a *function* of x, and right away we are leading into the terminology of Calculus. Recall the final sections of the preceding Chapter.

However, Calculus *really* begins when we start talking about the *rate of change* of y as x varies.

1 Rates of Change

One thing that is easy to "read off a graph" of $y(x)$ is the *slope* of the curve at any given point x. Now, if $y(x)$ is quite "curved" at the point of interest, it may seem contradictory to speak of its "slope," a property of a *straight* line. However, it is easy to see that as long as the curve is *smooth* it will always *look like a straight line* under sufficiently high *magnification*. This is illustrated in Fig. 1 for a typical $y(x)$ by a process of successive magnifications.

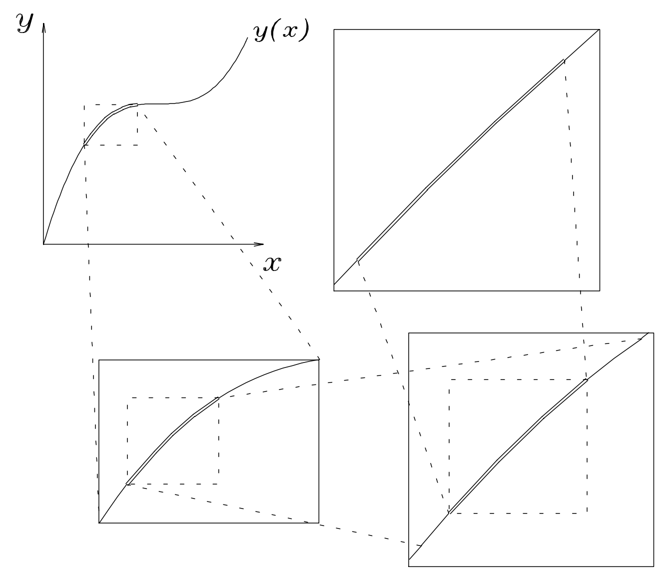

Figure 1 A series of "zooms" on a segment of the curve $y(x)$ showing how the *curved* line begins to look more and more like a *straight* line under higher and higher magnification.

We can also prescribe an algebraic method for *calculating* the slope, as illustrated in Fig. 2: the *definition* of the "slope" is the ratio of the increase in y to the increase in x on a vanishingly small interval. That is, when x goes from its initial value x_0 to a slightly larger value $x_0+\Delta x$, the curve carries y from its initial value $y_0 = y(x_0)$ to a new value $y_0+\Delta y = y(x_0+\Delta x)$, and the slope of the curve at $x = x_0$ is given by $\Delta y/\Delta x$ for a vanishingly small Δx. When a small change like Δx gets *really* small (*i.e.* small enough that the curve looks like a straight line on that interval, or "small enough to satisfy whatever criterion you want," then we write it differently, as dx, a "*differential*" (vanishingly small) change in x. Then the exact definition of the SLOPE of y with respect to x at some particular value of x, written in conventional Physics[1] language, is

$$\frac{dy}{dx} \equiv \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \equiv \lim_{\Delta x \to 0} \frac{y(x + \Delta x) - y(x)}{\Delta x} \quad (1)$$

This is best understood by an example: consider the simple function $y(x) = x^2$. Then

$$y(x + \Delta x) = (x + \Delta x)^2 = x^2 + 2x\Delta x + (\Delta x)^2$$

$$\text{and} \quad y(x + \Delta x) - y(x) = 2x\Delta x + (\Delta x)^2.$$

Divide this by Δx and we have

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x.$$

[1] Real Mathematicians prefer the "primed" notation, $dy/dx \equiv y'(x)$, for several reasons: first, it reminds us that dy/dx is also a *function of* x; the second reason will be obvious a little later. . . .

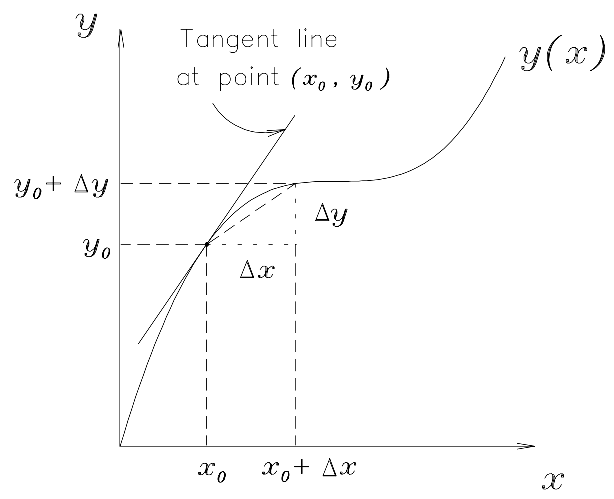

Figure 2 A graph of the function $y(x)$ showing how the *average* slope $\Delta y/\Delta x$ is obtained on a *finite* interval of the curve. By taking smaller and smaller intervals, one can eventually obtain the slope at a *point*, dy/dx.

Now let Δx shrink to zero, and all that remains is

$$\frac{\Delta y}{\Delta x} \xrightarrow[\Delta x \to 0]{} \frac{dy}{dx} = 2x.$$

Thus the slope [or *derivative*, as mathematicians are wont to call it] of $y(x) = x^2$ is $dy/dx = 2x$. That is, the slope increases linearly with x. The slope *of* the slope — which we call[2] the *curvature*, for obvious reasons — is then trivially $d(dy/dx)/dx \equiv d^2y/dx^2 = 2$, a constant. Make sure you can work this part out for yourself.

We have defined all these algebraic solutions to the geometrical problem of finding the slope of a curve on a graph in completely abstract terms — "x" and "y" indeed! What *are* x and y? Well, the whole idea is that they can be anything you want! The most common examples in Physics are when x is the *elapsed time*, usually written t, and y is the *distance travelled*, usually (alas) written x. Thus in an elementary Physics context the function you are apt to see used most often is $x(t)$, the position of some object as a function of time. This particular function has some very well-known derivatives, namely $dx/dt = v$, the *speed* or (as long as the motion is in a straight line!) *velocity* of the object; and $dv/dt \equiv d^2x/dt^2 = a$, the *acceleration* of the object. Note that both v and a are themselves (in general) functions of time: $v(t)$ and $a(t)$. This example so beautifully illustrates the "meaning" of the slope and curvature of a curve as first and second derivatives that many introductory Calculus courses and virtually all introductory Physics courses use it as *the* example to explain these Mathematical conventions. I just had to be different and start with something a little more formal, because I think you will find that the idea of one thing being a *function* of another thing, and the associated ideas of graphs and slopes and curvatures, are handy notions worth putting to work far from their traditional realm of classical kinematics.

[2]This differs from the conventional mathematical definition of *curvature*, $\kappa \equiv d\phi/ds$, where ϕ is the tangential angle and s is the arc length, but I like mine better, because it's simple, intuitive and useful. (OK, I'm a Philistine. So shoot me. ;-) Thanks to Mitchell Timin for pointing this out.

2 Second Derivatives

How about the rate of change **of** the rate of change? I slipped this in surreptitiously above when I defined the *curvature*,

$$\frac{d}{dx}\frac{dy}{dx} \equiv \frac{d^2y}{dx^2}$$

where the left hand side now explicitly displays the OPERATOR d/dx which means, "take the derivative with respect to x of whatever appears immediately to the right." (We will encounter other operators later on, so it's important to get used to this idea.)

In the prime Physics example where the vertical axis is *distance* and the horizontal axis is *time*, the *concave* graph corresponds to *acceleration* (speeding up of the speed) and the *convex* graph corresponds to *deceleration* (slowing down).

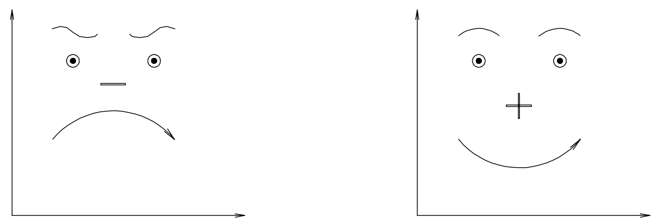

Figure 3 A graph of two functions, $y_-(x)$ [left] and $y_+(x)$ [right], having negative and positive curvature d^2y/dx^2, respectively. The frivolous cartoon format is an easy way to remember that a *negative* second derivative "curves downward" to make a *convex* "frowney face" whereas a *positive* second derivative "curves upward" to make a *concave* "smiley face".

3 Higher Derivatives

One can, of course, take the derivative of the derivative of the derivative,

$$\frac{d}{dt}\frac{d}{dt}\frac{dx}{dt} \equiv \frac{d^3x}{dt^3},$$

a.k.a. (in Physics) as the "*jerk*". (No, I'm not kidding.) In Physics we rarely go this far, because Newton's Second Law relates the *second* time derivative of distance (the *acceleration*) to the *mass* of a body and the *force* applied to it. But Mathematicians know no such restraint. They will happily refer to the n^{th} derivative,

$$\frac{d^ny}{dx^n}$$

which has the d/dx operator applied n times to $y(x)$. Later on we will encounter a function for which the n^{th} derivative of $y(x)$ is both nonzero and as simple as $y(x)$ itself — in fact, for which

$$\frac{d^ny}{dx^n} = y(x).$$

Stay tuned. . . [3]

[3] This would be a good time to remind you that Real Mathematicians prefer the notation $y'(x)$ instead of dy/dx. What do they use for the second derivative, d^2y/dx^2? Not surprisingly, they use $y''(x)$. For higher derivatives, I think the Physics notation d^5y/dx^5 is clearly preferable to the Mathematics notation $y'''''(x)$.

4 Integrals

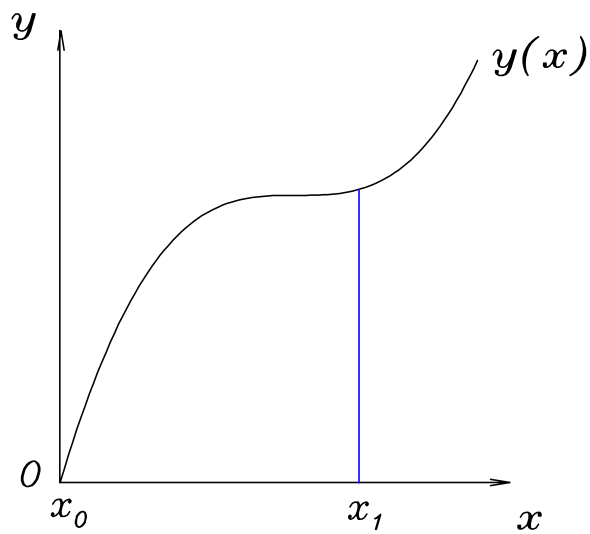

Figure 4 What is the **area under the curve** of $y(x)$?

Suppose that y is the number of new COVID-19 cases per day and x is time in units of days. We have all seen many curves like this in 2020. Then the total number of COVID-19 cases between day x_0 and day x_1 is given by

$$\int_{x_0}^{x_1} y(x)dx$$

(read "the integral of $y(x)$ with respect to x from x_0 to x_1"), whose rigorous, formal meaning is simply the **area under the curve** of $y(x)$ from x_0 to x_1.

The usual approach to evaluating this quantity is to break the area up into a large number of very skinny vertical rectangles of very narrow width Δx and height $y(x)$ and then let $\Delta x \to 0$ as the number of tall skinny rectangles becomes infinite. Although this formulation is easy to evaluate numerically on a computer, it does not lend itself to fun handwaving explanations that yield simple algebraic answers, so I'll be using the idea of *antiderivatives* — generally disdained by Real Mathematicians — to make it easier. Stay tuned.