Easy Derivatives

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1 Definition

Recall the definition of the derivative: the rate of change [slope] of a function at a point is the limiting value of its *average* slope over an interval including that point, as the width of the interval shrinks to zero:

$$\boxed{\frac{dy}{dx} \equiv \lim_{\Delta x \to 0} \frac{y(x+\Delta x) \; - \; y(x)}{\Delta x}}$$

All the remaining Laws and Rules can be proven by algebraic manipulation of this definition.

2 Operator Notation

The symbol $\frac{d}{dx}$ (read "derivative with respect to x") can be thought of as a mathematical "verb" (called an *operator*) which "operates on" whatever we place to its *right*. Thus[1]

$$\frac{d}{dx}[y] \equiv \frac{dy}{dx}$$

3 Product Rule

The derivative of the *product* of two functions is *not* the product of their derivatives! Instead,

$$\boxed{\frac{d}{dx}[f(x) \cdot g(x)] \; = \; \frac{df}{dx} \cdot g(x) \; + \; f(x) \cdot \frac{dg}{dx}}$$

Proof (Physicist's notation):

If $y(x) = f(x) \cdot g(x)$ then

$$y(x+\Delta x) = f(x+\Delta x) \cdot g(x+\Delta x)$$

$$= \left[f(x) + \frac{df}{dx}\Delta x\right]\left[g(x) + \frac{dg}{dx}\Delta x\right]$$

$$= f(x) \cdot g(x) + \left[\frac{df}{dx} \cdot g(x) + f(x) \cdot \frac{dg}{dx}\right]\Delta x$$

$$+ [\Delta x]^2 \frac{df}{dx} \cdot \frac{dg}{dx}$$

Divide this through by Δx and we have

$$\frac{y(x+\Delta x)}{\Delta x} = \frac{y(x)}{\Delta x} + \frac{df}{dx} \cdot g(x) + f(x) \cdot \frac{dg}{dx}$$

$$+ \Delta x \cdot \frac{df}{dx} \cdot \frac{dg}{dx}$$

Note that $y(x+\Delta x) - y(x) = \Delta y$ and let Δx shrink to zero, and all that remains is

$$\frac{\Delta y}{\Delta x} \underset{\Delta x \to 0}{\longrightarrow} \frac{dy}{dx} = \frac{df}{dx} \cdot g(x) + f(x) \cdot \frac{dg}{dx} \; . \; \mathcal{QED}$$

Now, you may find the expression for the change in $f(x)$,

$$\Delta f = \frac{df}{dx} \cdot \Delta x,$$

a little confusing: there's a Δx in the numerator and a dx in the denominator — which is which, and if we're going to make $\Delta x \to 0$ later, why not do it now and just cancel the two? We can't do that, and rather than try to explain why, I'll switch to Mathematician's notation, for the same reason *they* do!

Proof (Mathematician's notation):

If $y(x) = f(x) \cdot g(x)$ then

$$y(x+\Delta x) = f(x+\Delta x) \cdot g(x+\Delta x)$$

$$= [f(x) + f'(x) \cdot \Delta x]\,[g(x) + g'(x) \cdot \Delta x]$$

[1] I should take this opportunity to emphasize the difference between the "=" *equals* sign (meaning the thing on the left is the same *size* as the thing on the right) and the "≡" *equivalence* sign (meaning the thing on the left is by definition the *same thing* as that on the right). The latter is like the "==" *definition* operator in many programming languages.

$$= f(x) \cdot g(x) + [f'(x) \cdot g(x) + f(x) \cdot g'(x)]\,\Delta x$$
$$+[\Delta x]^2 f'(x) \cdot g'(x)$$

Divide this through by Δx and we have

$$\frac{y(x+\Delta x)}{\Delta x} = \frac{y(x)}{\Delta x} + f'(x) \cdot g(x) + f(x) \cdot g'(x)$$
$$+\Delta x \cdot f'(x) \cdot g'(x)$$

Note that $y(x+\Delta x) - y(x) = \Delta y$ and let Δx shrink to zero, and all that remains is

$$\boxed{\frac{\Delta y}{\Delta x} \underset{\Delta x \to 0}{\longrightarrow} y'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)\ .}$$

So it's true, whichever way you express it!

4 Examples

- **Constant times a Function:** Since the derivative of a *constant* is always *zero* (it *doesn't change*), the PRODUCT RULE gives

$$\frac{d}{dx}\left[a \cdot y(x)\right] \;=\; a \cdot \frac{dy}{dx}$$

  where a is any constant (*i.e.* not a function of x). This is sometimes referred to as "pulling the constant factor outside the derivative."

- **Power Law:** The simplest class of derivatives are those of power-law functions, $y(x) = x^p$. We have derived the result for $p = 2$ earlier; for $p = 3$ we have $y(x) = x \cdot x^2$, and since $dx/dx = 1$, the PRODUCT RULE gives

$$\frac{d}{dx}\left[x^3\right] \;=\; x \cdot 2\,x + 1 \cdot x^2 \;=\; 3x^2$$

  Using the same trick, you can easily show that $dy/dx = 4x^3$ for $y(x) = x^4$, $dy/dx = 5x^4$ for $y(x) = x^5$, and so on for all integer values of p. It turns out that the general result

$$\boxed{\frac{d}{dx}\left[x^p\right] \;=\; p\,x^{p-1}}$$

  is valid for *all* powers p, whether positive, negative, integer, rational, irrational, real, imaginary or complex. That's a little harder to prove, but you can look it up on *Wikipedia*.

- **Function of a Function:** Suppose y is a function of x and x is in turn a function of t. Then if t changes by Δt, x changes by

$$\Delta x = \frac{dx}{dt} \cdot \Delta t$$

  and y changes by

$$\Delta y = \frac{dy}{dx} \cdot \Delta x = \frac{dy}{dx} \cdot \frac{dx}{dt} \cdot \Delta t.$$

  Dividing both sides by Δt gives

$$\frac{\Delta y}{\Delta t} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

  and if we let $\Delta t \to 0$ we get

$$\boxed{\frac{d}{dt}\left\{y[x(t)]\right\} \;=\; \frac{dy}{dx} \cdot \frac{dx}{dt}}$$

  (CHAIN RULE)