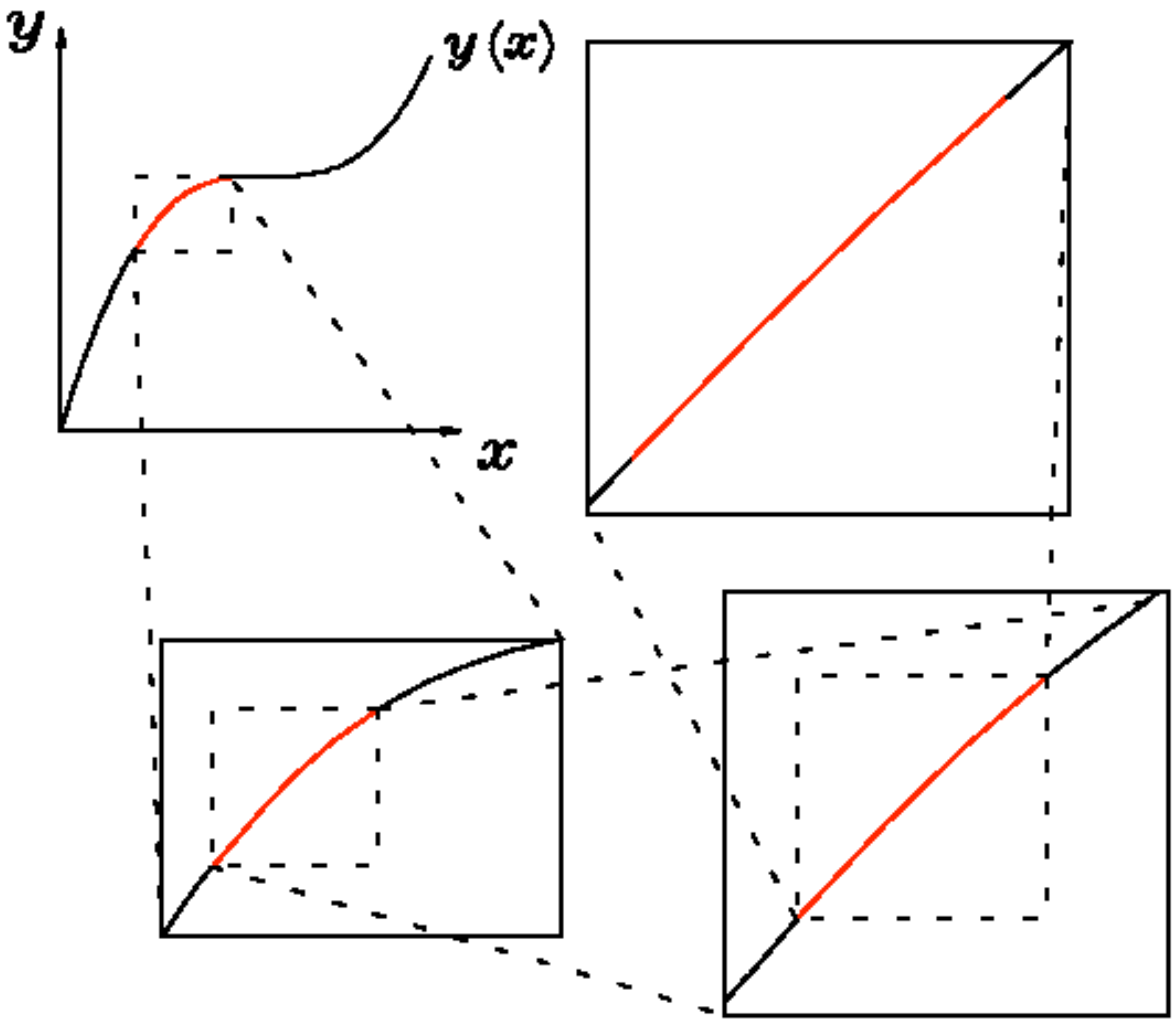
Easy Calculus

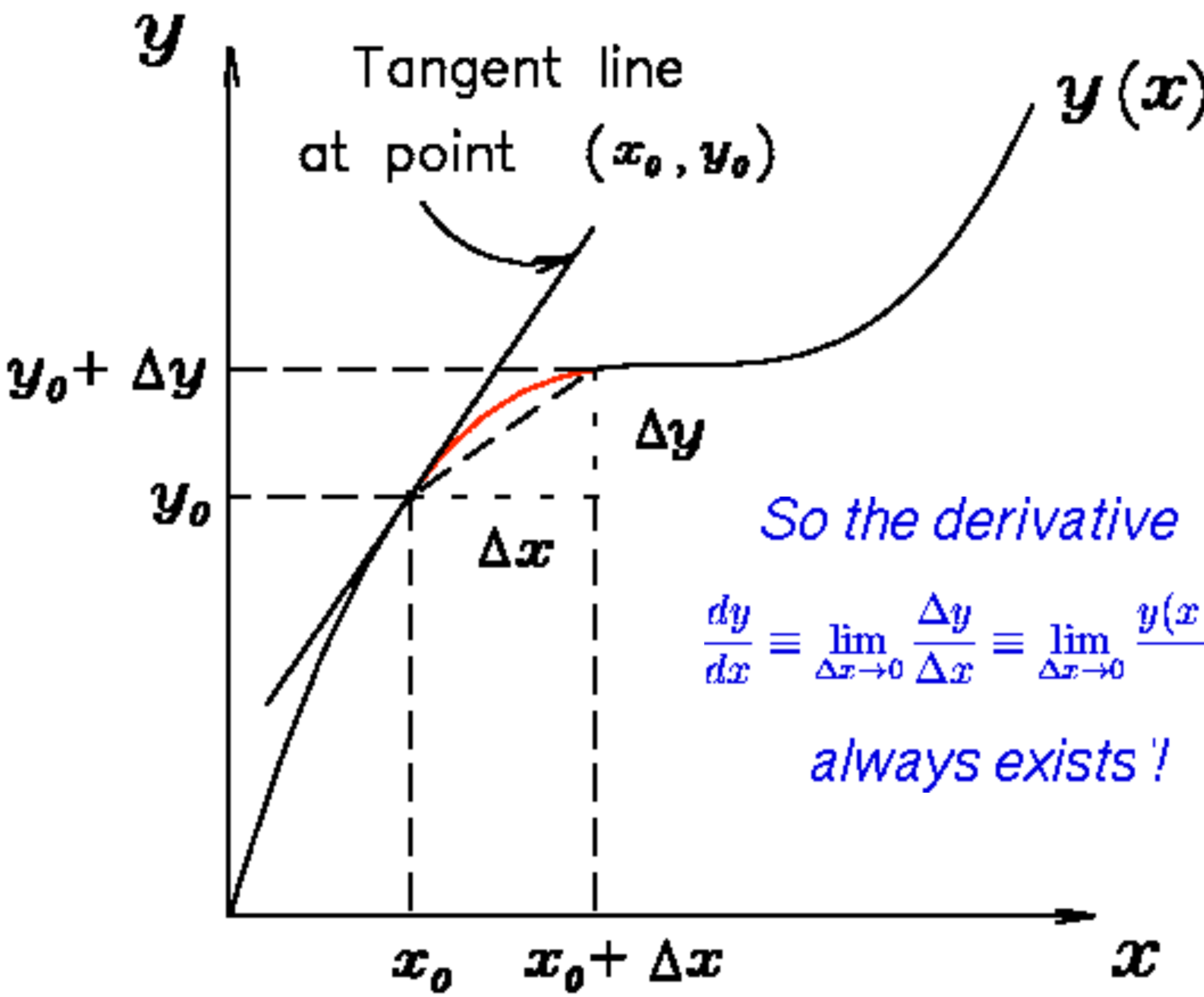
a Hand-Waver's Guide

(blame ***Jess***)

Rule 1: A curved line looks straight if you blow it up enough!



Rule 2: There are ^probably **no discontinuities** in the real, physical world.



So the derivative

$$\frac{dy}{dx} \equiv \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \equiv \lim_{\Delta x \to 0} \frac{y(x+\Delta x) - y(x)}{\Delta x}$$

always exists !

A few easy-to-remember derivatives:

Power Law:

$$\frac{d}{dx}(x^p) = p\,x^{p-1}$$

$(p \neq 0)$

Constant × a Function:

$$\frac{d}{dx}[a\,y(x)] = a\,\frac{dy}{dx}$$

$(a = \text{const})$

Product Law:

$$\frac{d}{dx}[f(x) \cdot g(x)] = \frac{df}{dx} \cdot g(x) + f(x) \cdot \frac{dg}{dx}$$

Chain Rule:

$$\frac{d}{dt}y[x(t)] = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

...but don't take my word for it!

Deriving the derivatives:

Operator Notation:

$$\frac{d}{dx}[y] \equiv \frac{dy}{dx}$$

Mathematician's Notation:

$$\frac{dy}{dx} \equiv y'(x)$$

For Small Changes Δx:

$$\Delta y = y'(x)\Delta x$$

Deriving the Product Law:

If $y(x) = f(x) \cdot g(x)$ then

$$y(x + \Delta x) = f(x + \Delta x) \cdot g(x + \Delta x)$$

$$= [f(x) + f'(x) \cdot \Delta x] [g(x) + g'(x) \cdot \Delta x]$$

$$= f(x) \cdot g(x) + [f'(x) \cdot g(x) + f(x) \cdot g'(x)] \Delta x$$

$$+ [\Delta x]^2 f'(x) \cdot g'(x)$$

Divide this through by Δx and we have

$$\frac{y(x + \Delta x)}{\Delta x} = \frac{y(x)}{\Delta x} + f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

$$+ \Delta x \cdot f'(x) \cdot g'(x)$$

Note that $y(x + \Delta x) - y(x) \models \Delta y$ and let Δx shrink to zero, and all that remains is

$$\boxed{\frac{\Delta y}{\Delta x} \underset{\Delta x \to 0}{\longrightarrow} y'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x) \ .}$$

Constant times a Function:

$$\frac{d}{dx}\left[a \cdot y(x)\right] \; = \; a \cdot \frac{dy}{dx}$$

Power Law:

$$p = 1: \quad \frac{d}{dx}[x] = \frac{dx}{dx} = 1$$

$$p = 2: \quad \frac{d}{dx}[x^2] = \frac{d}{dx}[x \cdot x] = 1 \cdot x + x \cdot 1 = 2x$$

$$p = 3: \quad \frac{d}{dx}[x^3] = \frac{d}{dx}[x \cdot x^2] = 1 \cdot x^2 + x \cdot 2x = 3x^2$$

$$\text{General}: \quad \frac{d}{dx}[x^p] = p\, x^{p-1}$$

Deriving the Chain Rule:

Function of a Function: Suppose y is a function of x and x is in turn a function of t. Then if t changes by Δt, x changes by

$$\Delta x = \frac{dx}{dt} \cdot \Delta t$$

and y changes by

$$\Delta y = \frac{dy}{dx} \cdot \Delta x = \frac{dy}{dx} \cdot \frac{dx}{dt} \cdot \Delta t.$$

Dividing both sides by Δt gives

$$\frac{\Delta y}{\Delta t} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

and if we let $\Delta t \rightarrow 0$ we get

$$\boxed{\frac{d}{dt}\{y[x(t)]\} = \frac{dy}{dx} \cdot \frac{dx}{dt}}$$

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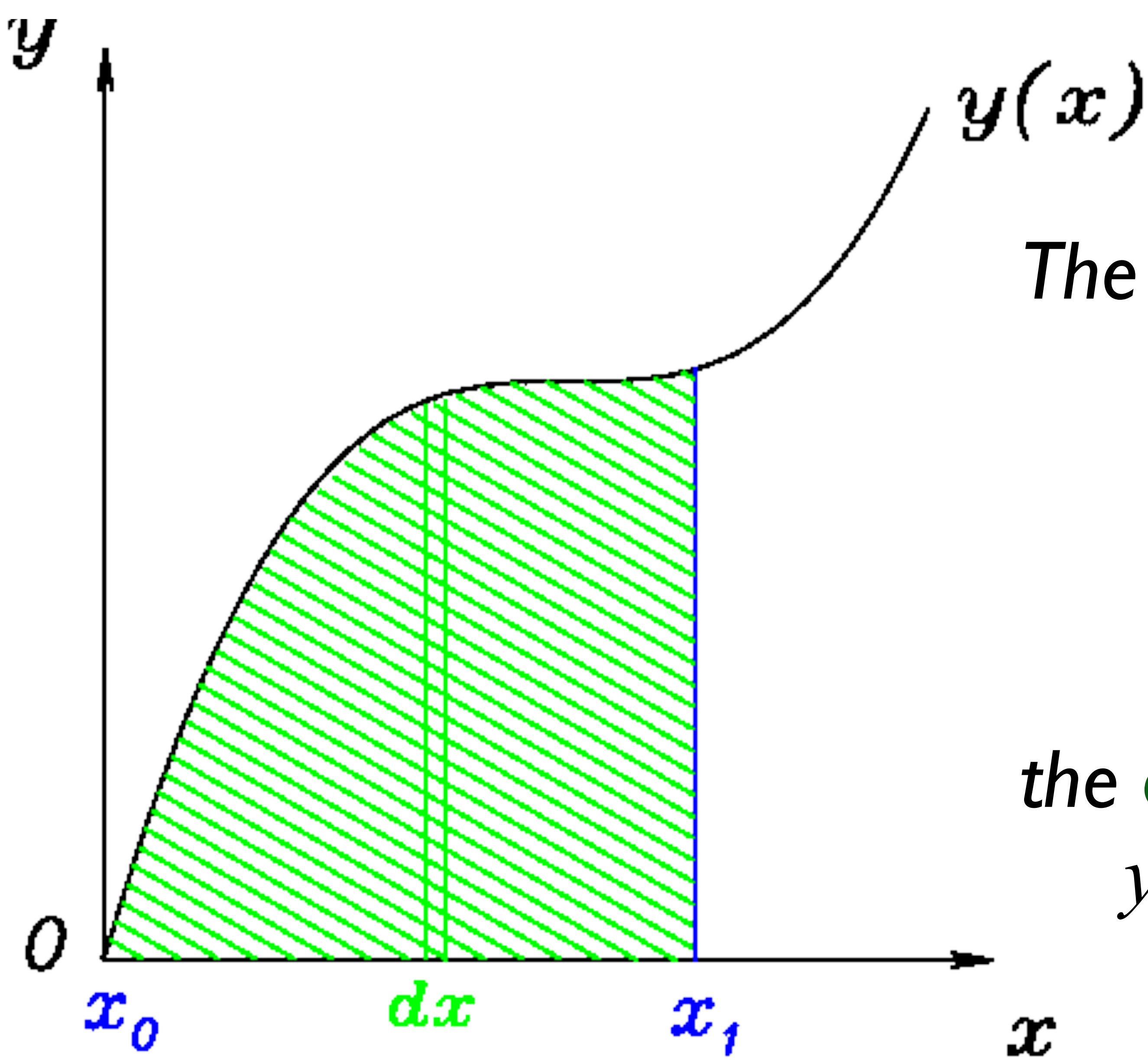
INTEGRALS



The **Definite Integral**

$$\int_{x_0}^{x_1} y(x)dx$$

is *defined* as

the ***area under the curve***

$y(x)$ between x_0 and x_1.

It is described in terms of adding up many vertical "slices" of infinitesimal width dx and height $y(x)$.

INTEGRALS

The ***Indefinite Integral*** *(a.k.a.* ***Antiderivative****) of* $y(x)$ *is better thought of as* *the function whose derivative is* $y(x)$.

Just ask,

"What Function Has This Derivative?"

If $g(x) = a = df/dx$, what is $f(x)$?

Answer: $f(x) = \int a\,dx = a\,x$ + const.

If $g(x) = 2\,b\,x = df/dx$, what is $f(x)$?

Answer: $f(x) = \int 2\,b\,x\,dx = b\,x^2$ + const.

More Examples

of ***Indefinite Integrals*** (***Antiderivatives***)

if $g(x) = x^n = df/dx$, what is $f(x)$?

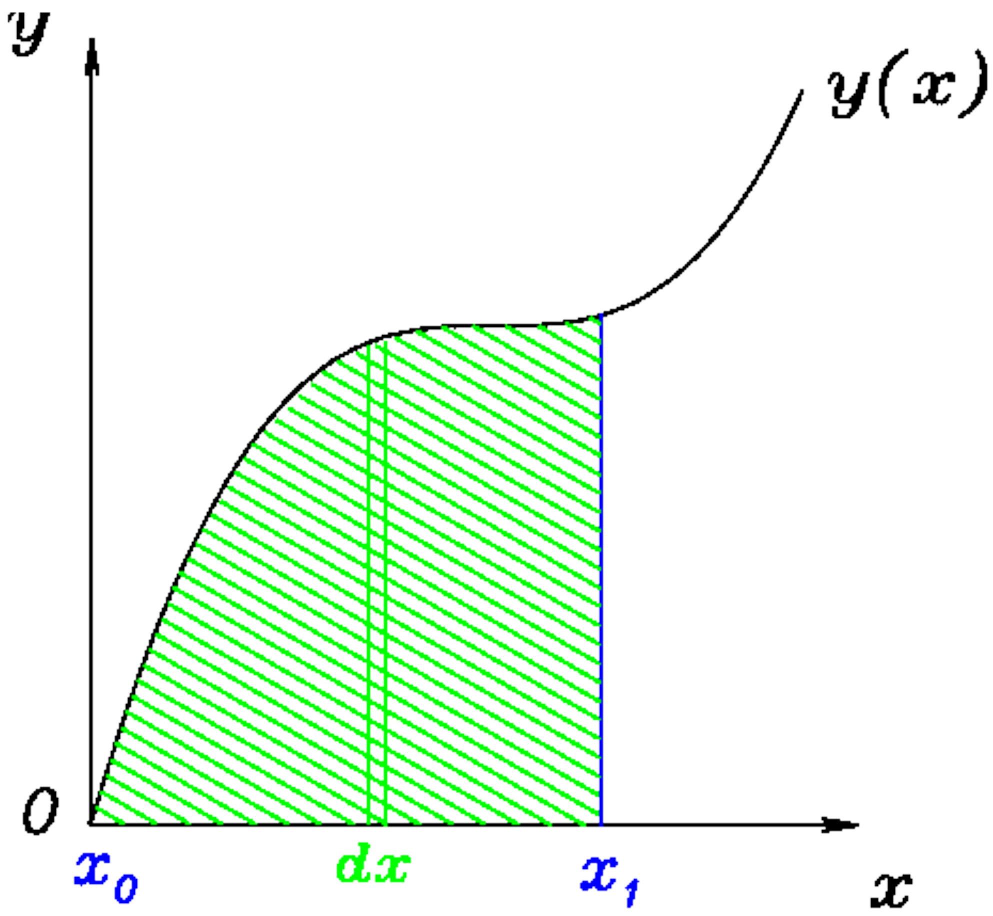
Answer: $f(x) = \int x^n \, dx = \dfrac{x^{n+1}}{n+1}$ + const.

if $g(x) = 1/x^n \equiv x^{-n} = df/dx$, what is $f(x)$?

Answer: $f(x) = \int x^{-n} \, dx = \dfrac{x^{-n+1}}{-n+1}$ + const.

Try this: if $g(x) = 1/x \equiv x^{-1} = df/dx$, *what is* $f(x)$ *?*

INTEGRALS



The **Definite Integral**

$$\int_{x_0}^{x_1} y(x)dx$$

is defined as

the **area under the curve**

$y(x)$ between x_0 and x_1.

But it is equal to the **difference** between the **Antiderivative** $f(x)$ of $y(x)$ at the endpoints: .

$$\int_{x_0}^{x_1} y(x)dx = f(x_1) - f(x_0)$$

Examples

of ***Definite Integrals***

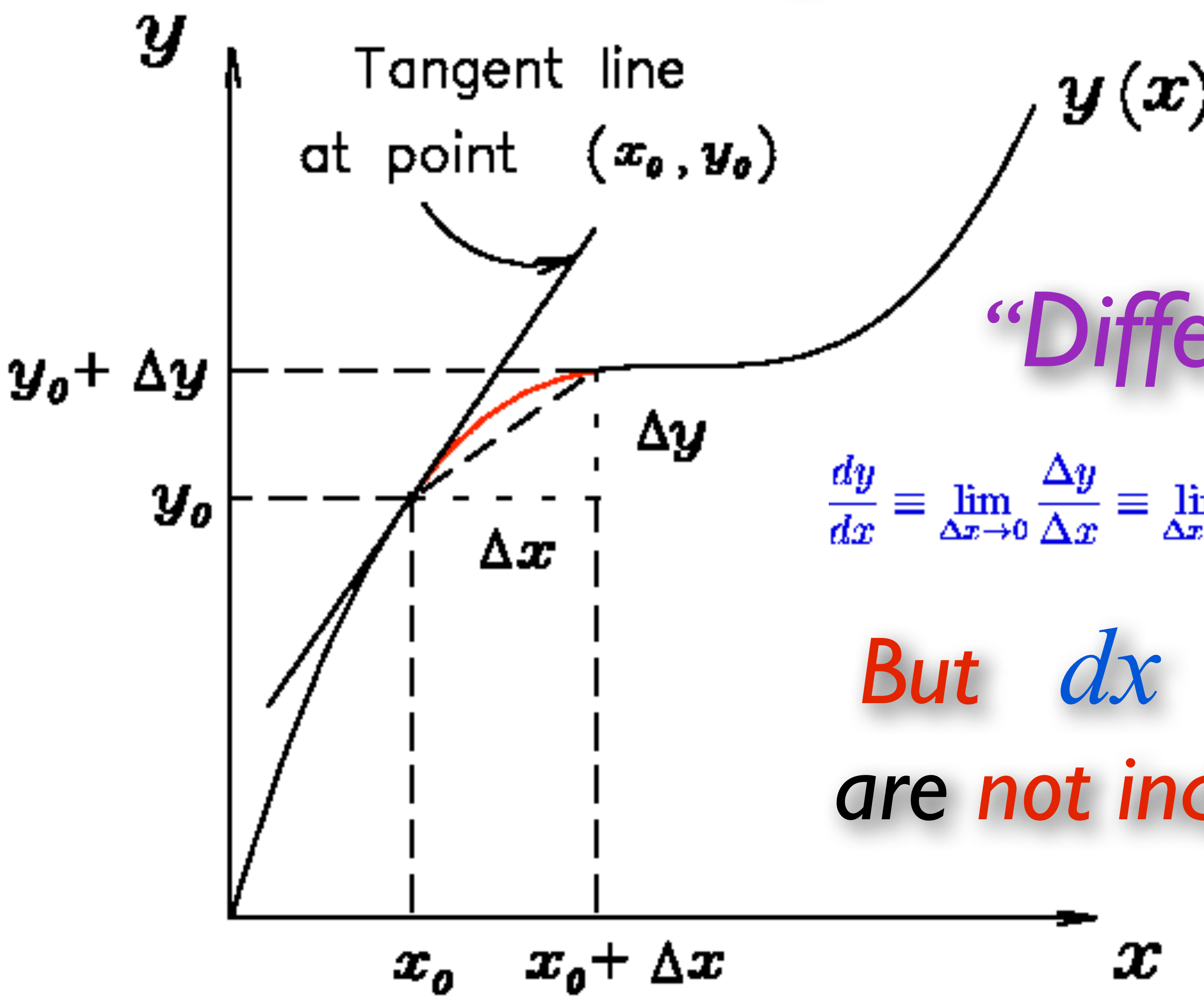
If $y(x) = x^2$, $x_0 = 1$ and $x_1 = 2$, what is $\int_{x_0}^{x_1} y(x)dx$?

Answer: $\dfrac{x_1^3 - x_0^3}{3} = \dfrac{8 - 1}{3} = 7/3$

If $y(x) = 1/x^2 \equiv x^{-2}$, $x_0 = 2$ and $x_1 = 4$, what is $\int_{x_0}^{x_1} y(x)dx$?

Answer: $\dfrac{x_1^{-1} - x_0^{-1}}{-1} = \dfrac{¼ - ½}{-1} = ¼$

Rule 3: dx is just a really, **really small** Δx



"Differentials"

$$\frac{dy}{dx} \equiv \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \equiv \lim_{\Delta x \to 0} \frac{y(x + \Delta x) - y(x)}{\Delta x}$$

But dx and dy are *not independent!*

Rule 4: *If we're really,* ***really*** *careful and never forget that*

dv, dx and dt

are not independent,

We can do algebra with Differentials !

Momentum & Impulse

$$F = m\,a \;\;\&\;\; a \equiv dv/dt \;\Rightarrow\; \boxed{m\,dv = F\,dt}$$

$$a \equiv dv/dt \;\;\&\;\; v \equiv dx/dt \;\Rightarrow\; v\,dv/dt = a\,dx/dt$$

Kinetic Energy & Work

Cancel dt's & add $F = m\,a \;\Rightarrow\; \boxed{mv\,dv = F\,dx}$

So What?

Change in Momentum $p = m\,v$

= *Impulse* $\int F(t)\,dt$

(Useful when we know the force as a function of time.)

Change in Kinetic Energy $K = \frac{1}{2}\,m\,v^2$

= *Work* $\int F(x)\,dx$

(Useful when we know the force as a function of position.)