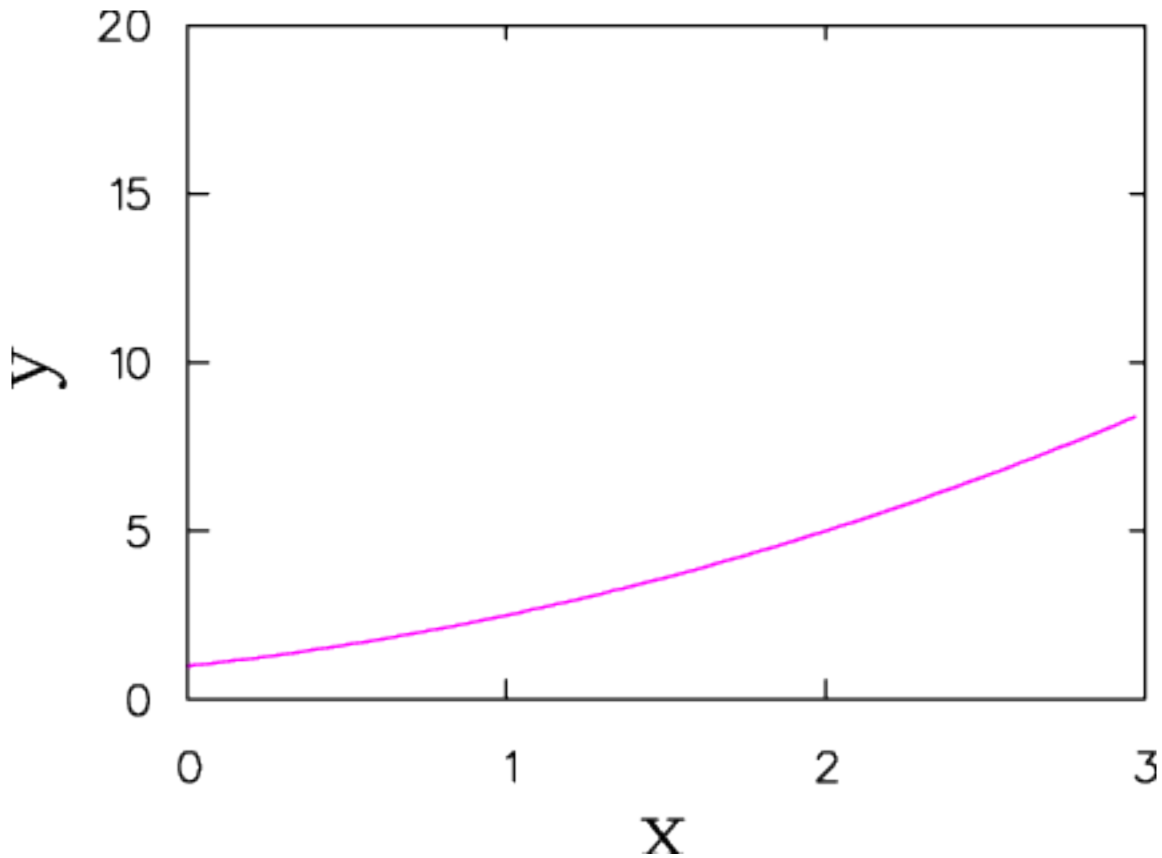
Exponential

GROWTH and **DECAY**

Guess the Function:



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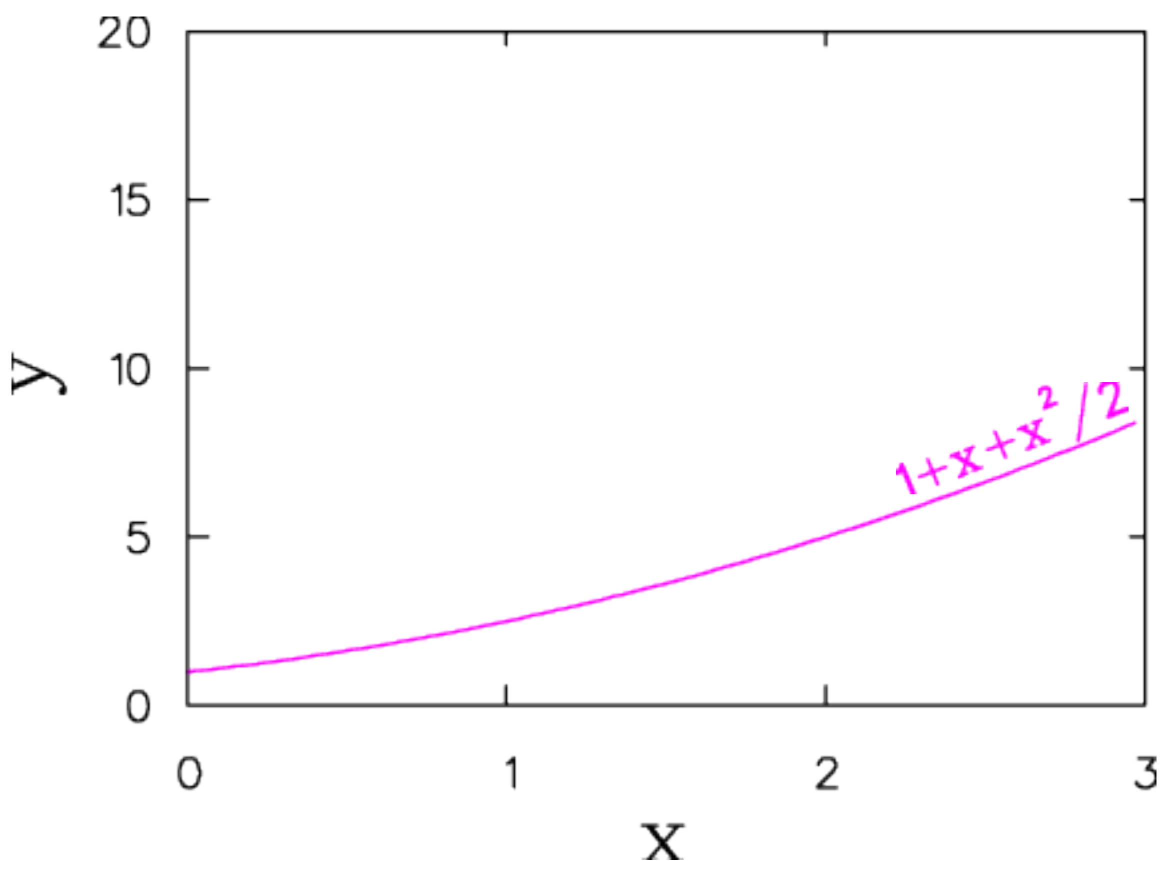

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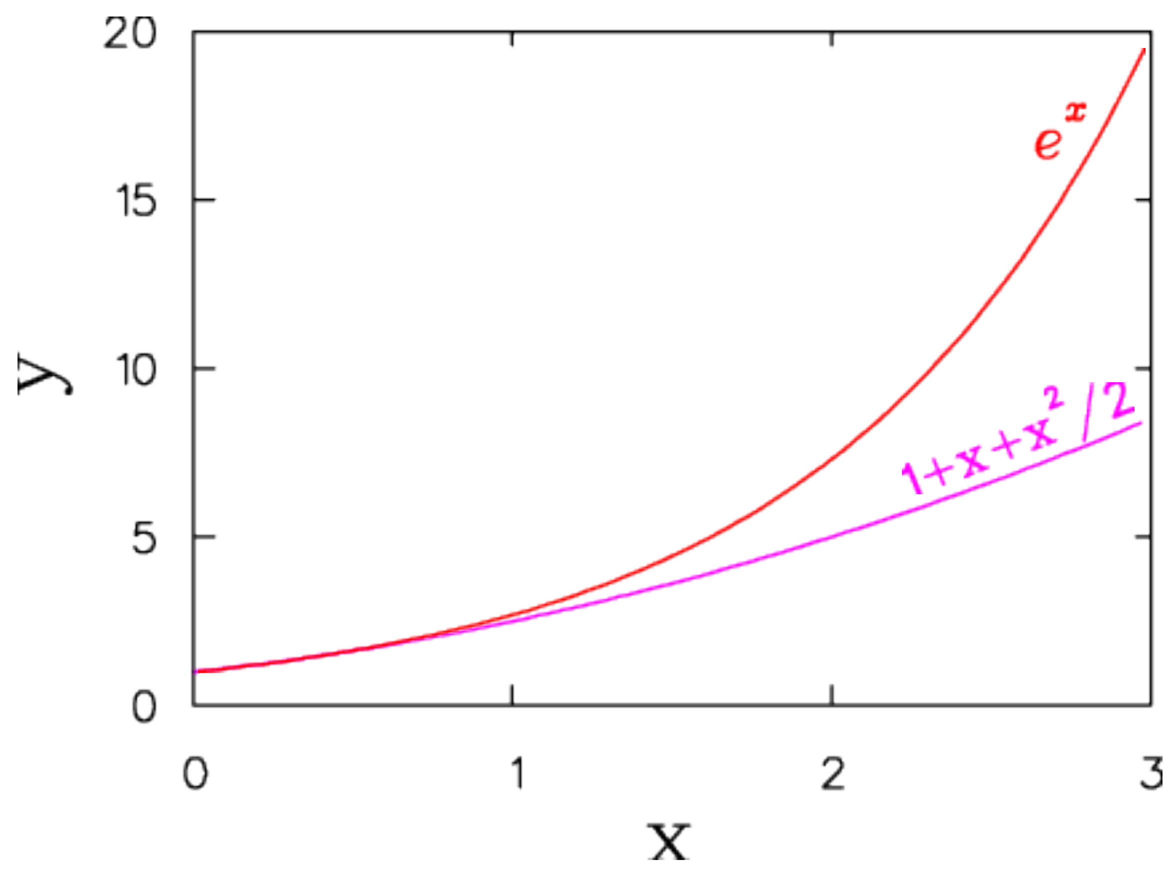

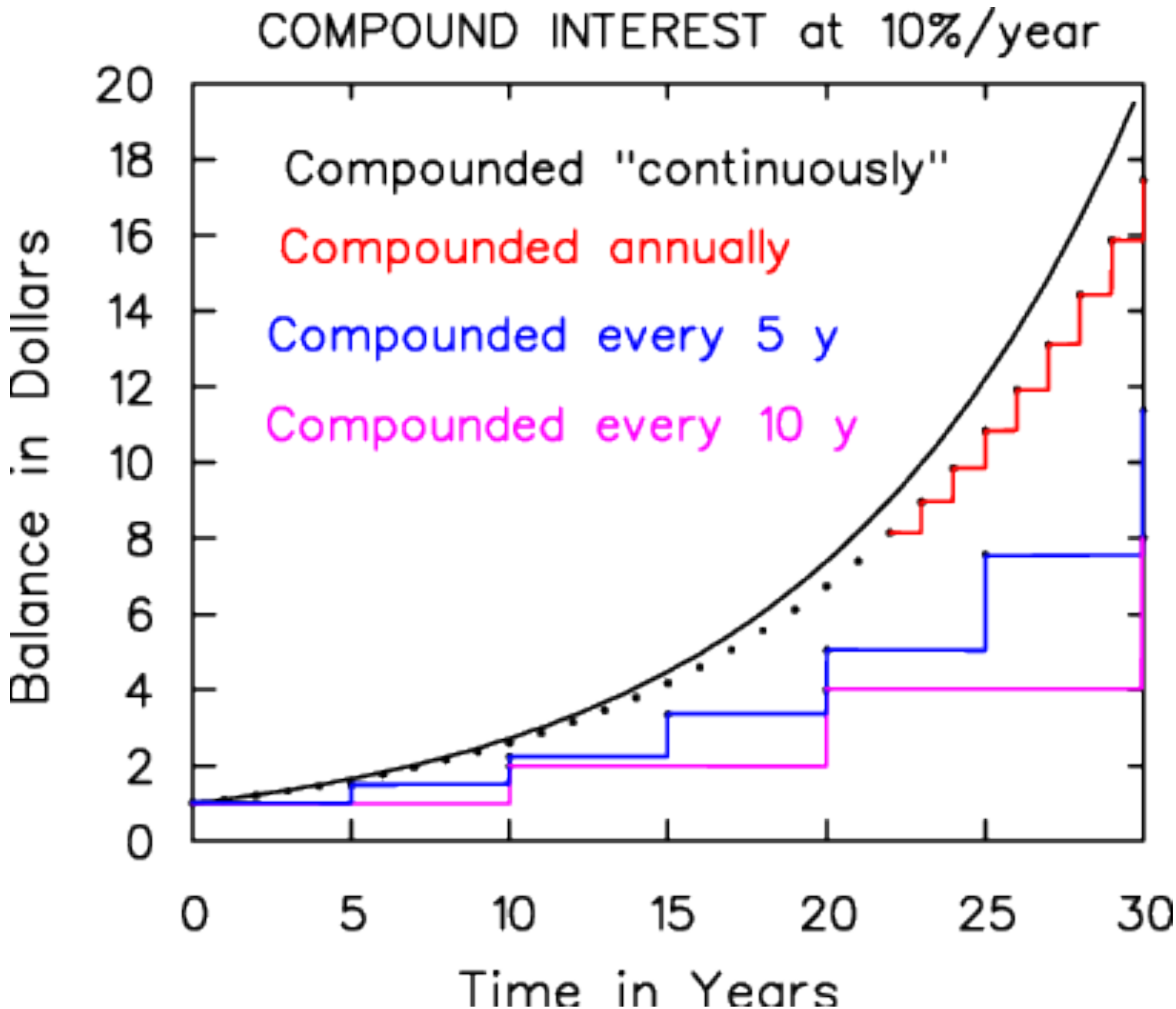
COMPOUND INTEREST at 10%/year
Compounded "continuously"
Compounded annually
Compounded every 5 y
Compounded every 10 y
Balance in Dollars
Time in Years
0
2
4
6
8
10
12
14
16
18
20
0
5
10
15
20
25
30

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- What if this were still true as $\Delta t \to 0$? $\mathrm{d}B/\mathrm{d}t = 0.1\, B$
- Or, more generally, $\mathrm{d}B/\mathrm{d}t = k\, B$ where k is in *inverse time* units.

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$$B(t) \;=\; a_0 \;+\; a_1\,t \;+\; a_2\,t^2 \;+\; a_3\,t^3 \;+\; \ldots$$

Let's check!

Hypothesis: $B(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots$

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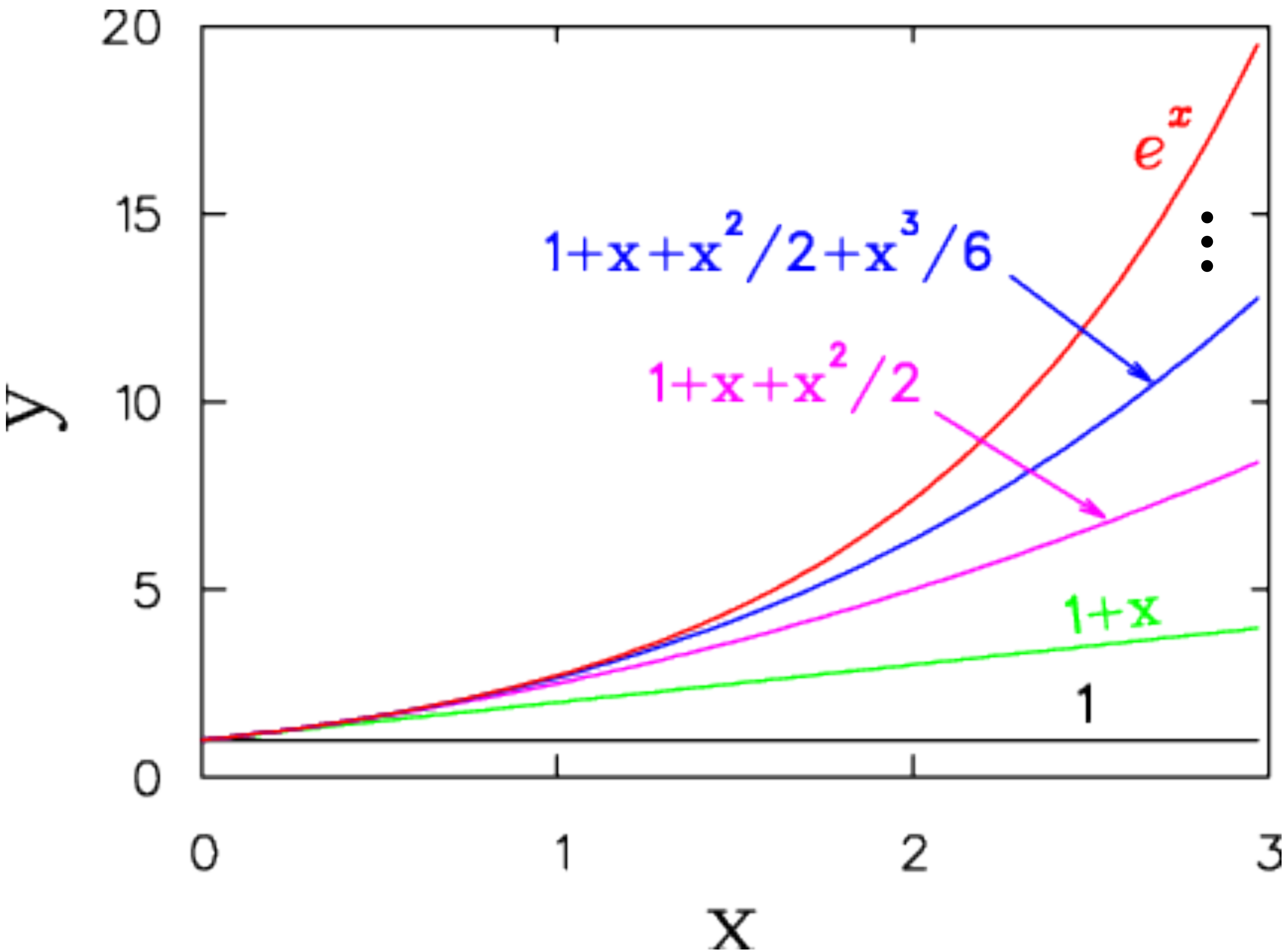
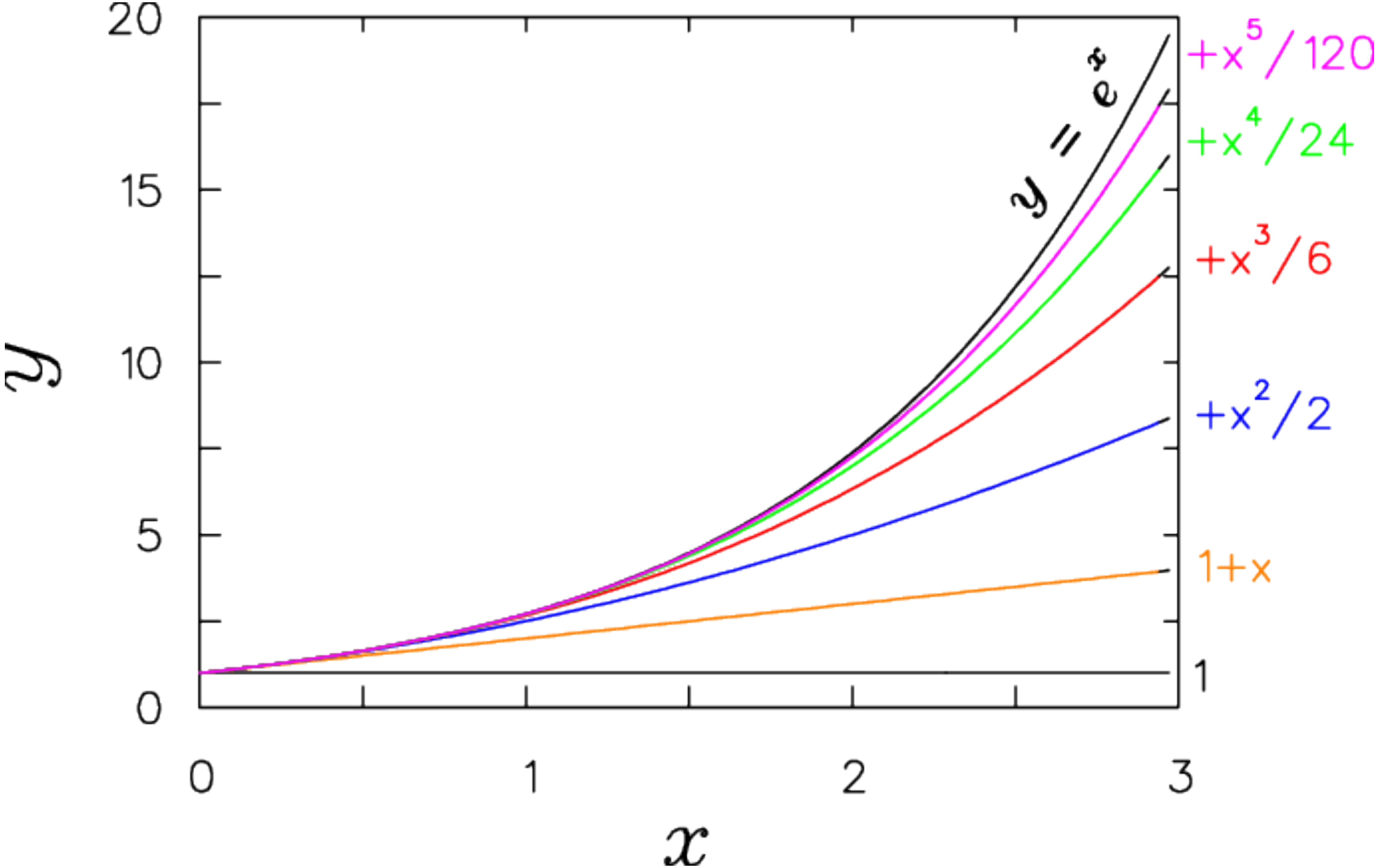
GRAPHICALLY :



Another View:



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(Euler's Theorem)

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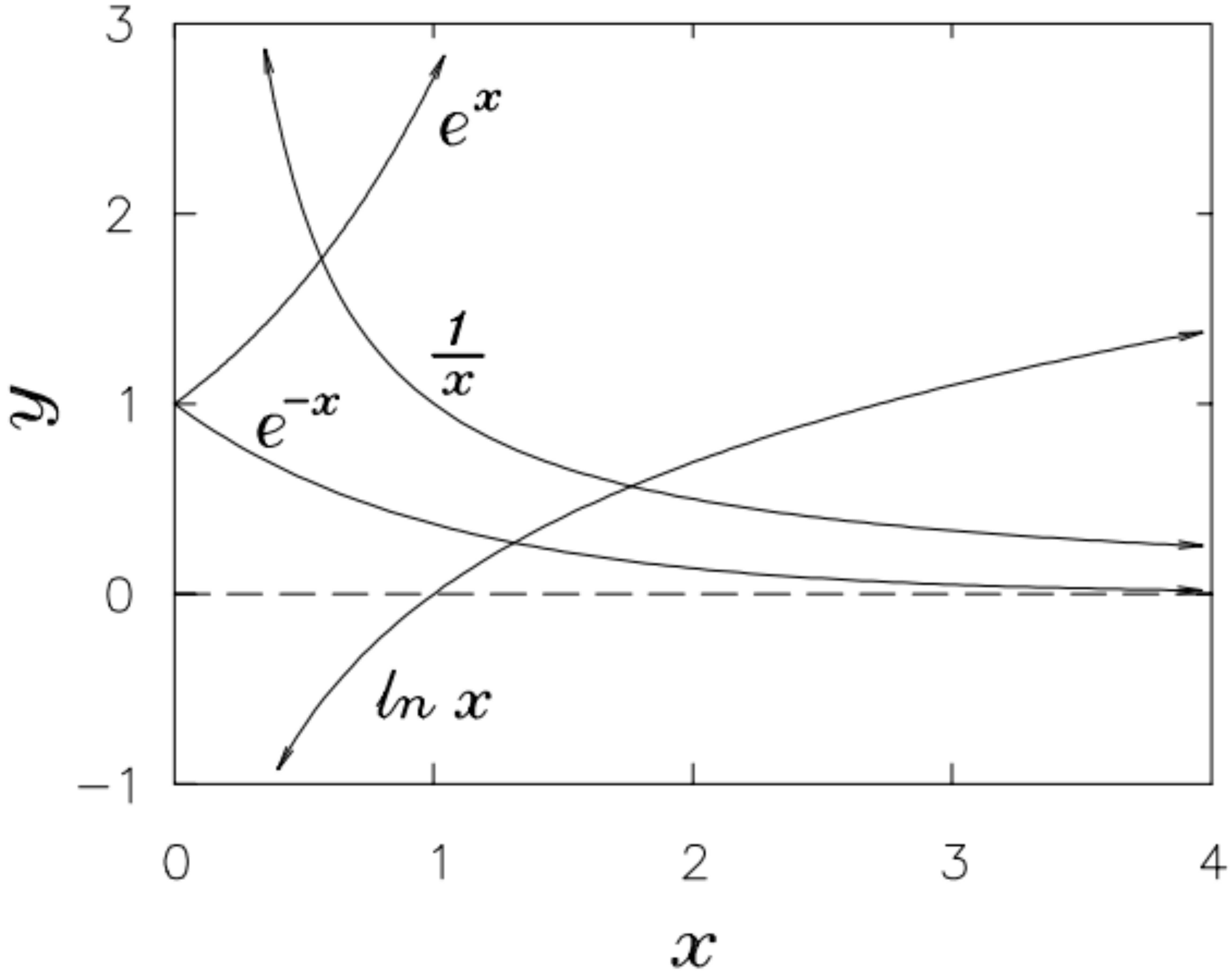
So if $y(x) = 1/x \equiv x^{-1}$, $\displaystyle\int_{x_0}^{x_1} y(x)dx = \ell n(x_1/x_0)$



3
2
1
0
−1
0
1
2
3
4
y
x
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$\frac{1}{x}$
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- Radioactive decay: $\lambda = 1/\tau = \ell n 2 / T_{½}$ $(\ell n 2 = 0.6931478\ldots)$

- Complex exponentials: if $\kappa = -\gamma + i\omega$,
  $e^{\kappa t} = e^{-\gamma t} (\cos \omega t + i \sin \omega t)$ [damped oscillations]

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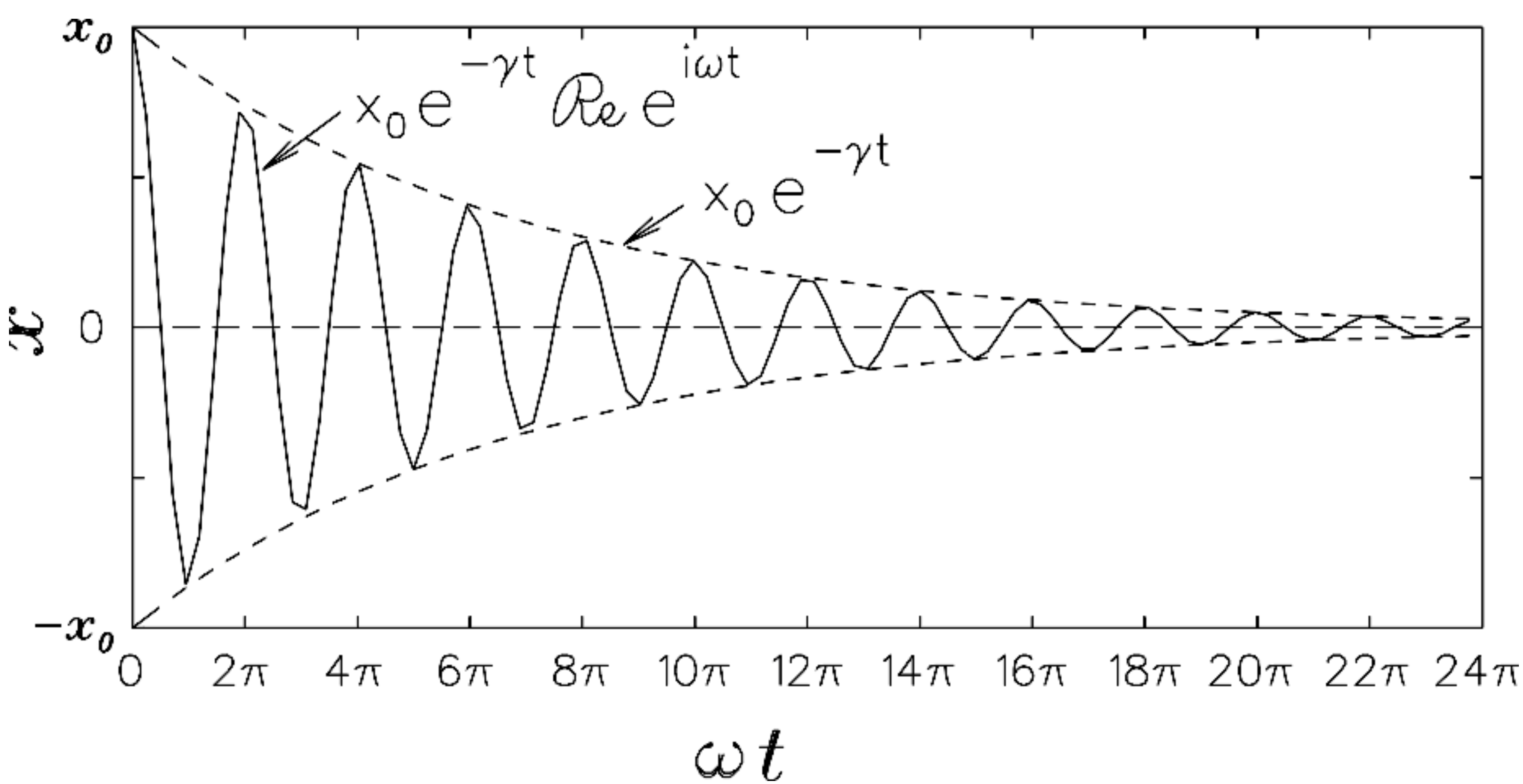

Finis