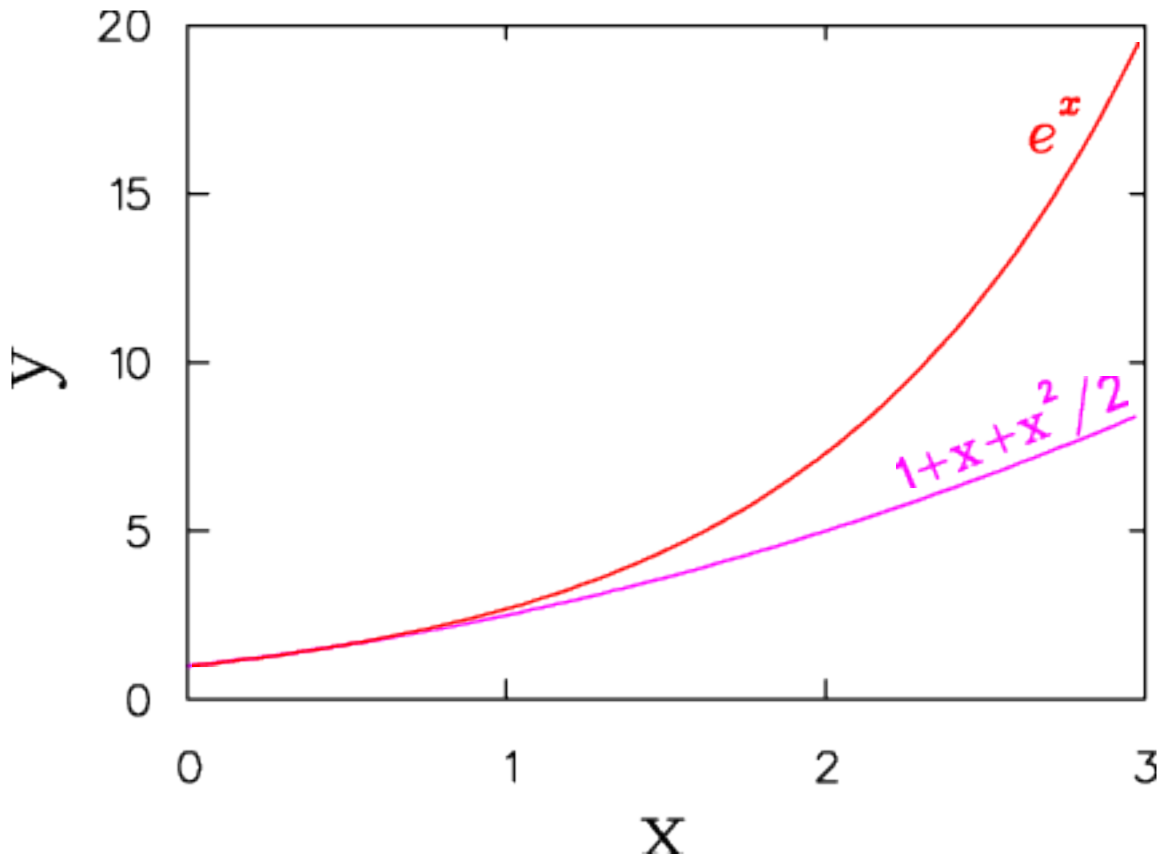
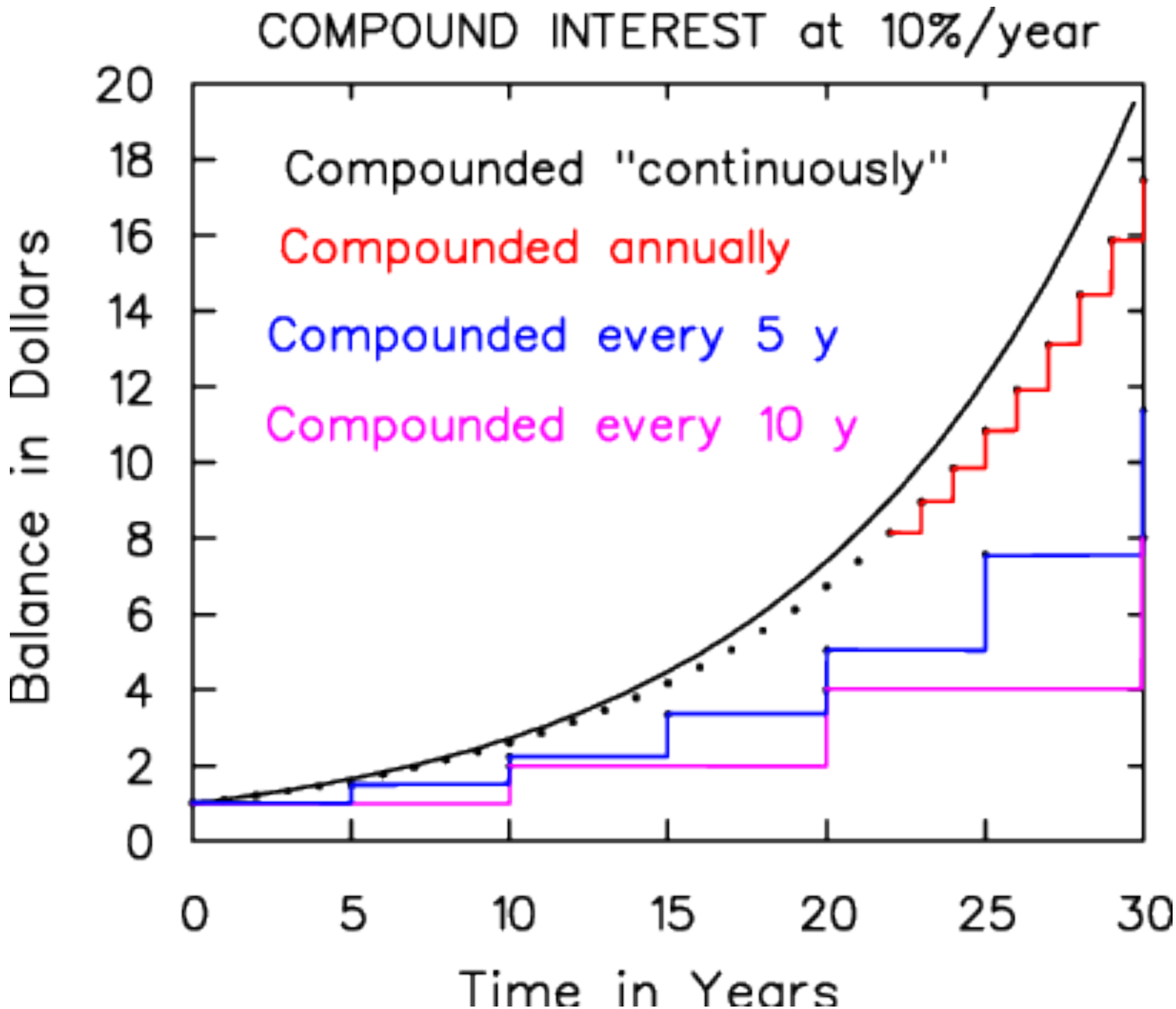
Exponential

GROWTH and **DECAY**

Guess the Function:





COMPOUND INTEREST at 10%/year
Compounded "continuously"
Compounded annually
Compounded every 5 y
Compounded every 10 y
Balance in Dollars
Time in Years
0
2
4
6
8
10
12
14
16
18
20
0
5
10
15
20
25
30

Finite *vs*. Infinitesimal Differences

- Let B be the balance in your savings account.
- Let t be the elapsed time in years.
- Let the **function** $B(t)$ be the ***recipe*** for how B changes with t :
- After Δt = 1 year, $B(t + \Delta t) = B(t) + 0.1\, B(t) = B(t) + \Delta B$
- Thus $\Delta B/\Delta t = 0.1\, B$
- What if this were still true as $\Delta t \to 0$? $\mathrm{d}B/\mathrm{d}t = 0.1\, B$
- Or, more generally, $\mathrm{d}B/\mathrm{d}t = k\, B$ where k is in *inverse time* units.

What is this *simpler* function?

$$\mathrm{d}B/\mathrm{d}t = B \qquad (\textit{i.e.}\ \ k = 1)$$

(B is *its own derivative*!)

Then it's also its own *second* derivative… and *third* derivative… and n^{th} derivative:

$$\mathrm{d}^2B/\mathrm{d}t^2 = \mathrm{d}^3B/\mathrm{d}t^3 = \mathrm{d}^nB/\mathrm{d}t^n = B$$

Can we express $B(t)$ as a simple *polynomial*?

$$B(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots$$

Let's check!

Hypothesis: $B(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots$

Defining Condition: $\mathrm{d}B/\mathrm{d}t = B$

(B is *its own derivative*)

Initial Condition: $B = 1$ at $t = 0 \Rightarrow a_0 = 1$

Differentiate: $\mathrm{d}B/\mathrm{d}t = 0 + a_1 + 2a_2 t + 3a_3 t^2 + \ldots$

$= B = 1 + a_1 t + a_2 t^2$

$+ a_3 t^3 + \ldots$

For this to be true, we need $2a_2 = a_1 = 1$ or $a_2 = ½$

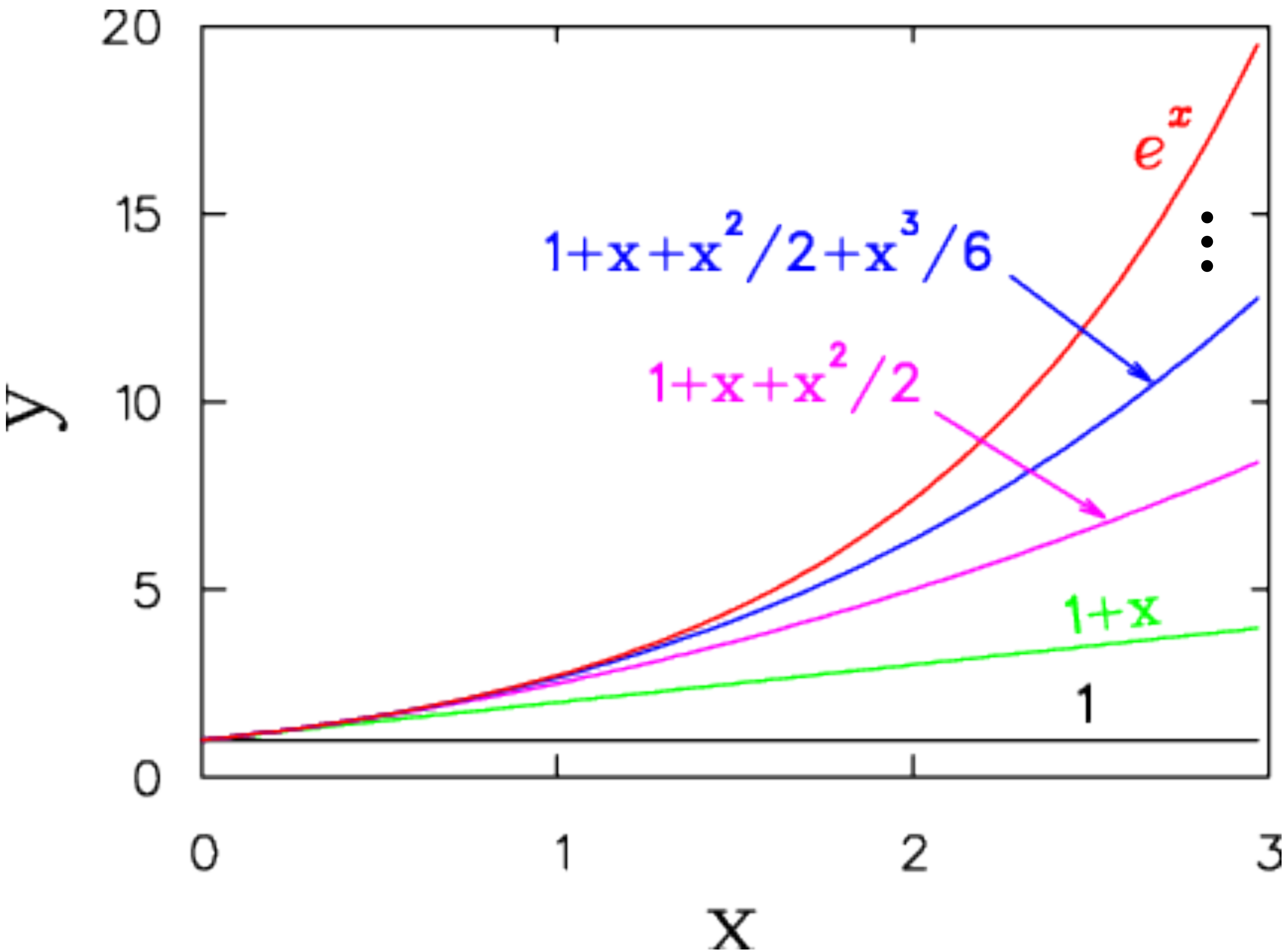
and $3a_3 = a_2 = ½$ or $a_3 = 1/(2\times3)$ and so on…

$$B(t) = \sum_{n=0}^{\infty} t^n/n! \equiv \exp(t)$$

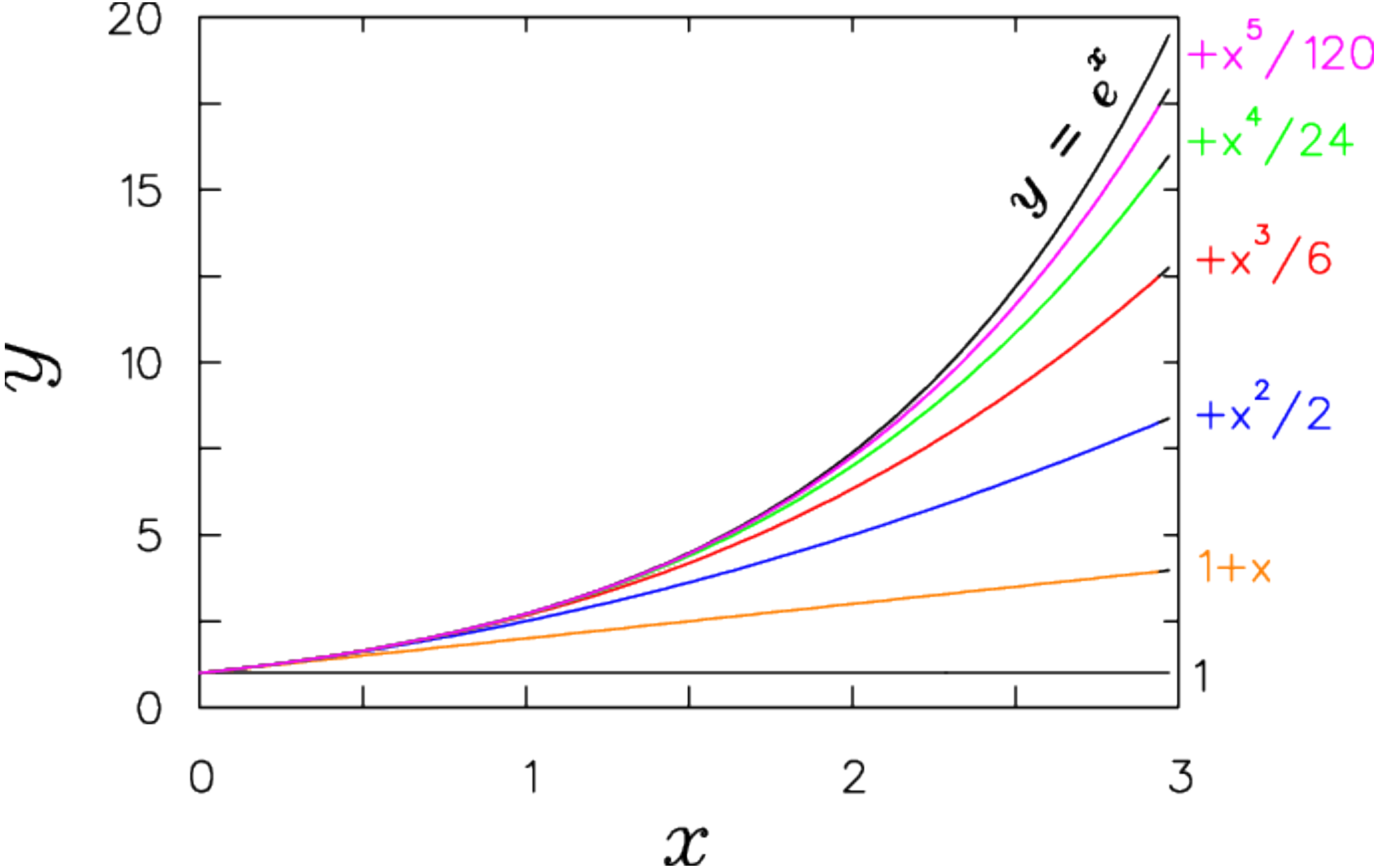
$(0! = 1)$

where $n! \equiv n\,(n\text{-}1)\,(n\text{-}2)\,\ldots\,(3)(2)(1)$.

GRAPHICALLY :



Another View:



Properties of the Exponential Function

$$\exp(t) = \sum_{n=0}^{\infty} t^n/n!$$

- It *grows* faster than *any* power law!
- $\exp(1) = 1 + 1 + ½ + 1/6 + 1/24 + \ldots \equiv e = 2.718281828459045\ldots$
- It can be written as e raised to the t power: $\exp(t) \equiv e^t$
- $e^{a+b} = e^a \times e^b$ just as (*e.g.*) $x^{2+3} = x^2 \times x^3 = x^5$
- $\exp(-t) \equiv e^{-t} \equiv 1/e^t$ *shrinks* faster than any *inverse* power law.
- $e^{it} = 1 + i\,t - ½\,t^2 - (1/6)\,i\,t^3 + (1/24)\,t^4\ldots = \cos t + i \sin t$

(Euler’s Theorem)

Natural Logarithm $\ell n(x)$:

The *Inverse* of the Exponential Function

(the **power** to which one must raise e to obtain x)

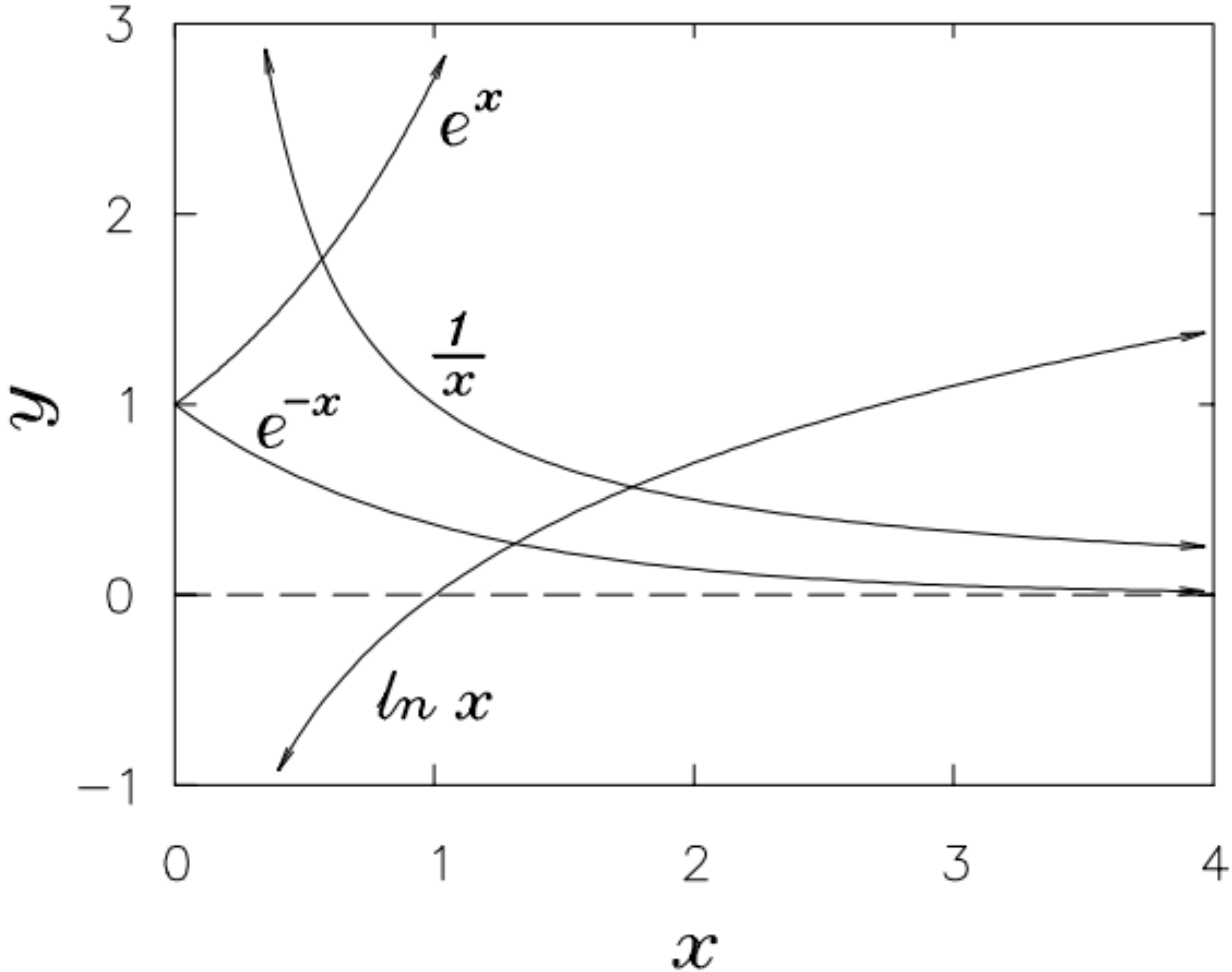
$$e^{\ell n(x)} = x$$

By the same token,

$$\ell n(e^x) = x$$

SWOP: if $y(x) = \ell n\,(x)$, $\mathrm{d}y/\mathrm{d}x = 1/x \equiv x^{-1}$

So if $y(x) = 1/x \equiv x^{-1}$, $\int_{x_0}^{x_1} y(x)dx = \ell n(x_1/x_0)$



3
2
1
0
−1
0
1
2
3
4
e^x
$\frac{1}{x}$
e^{-x}
$\ln x$
y
x

Applications of the Exponential Function e^{kx} and its siblings $e^{-\lambda x}$ & $\ell n(x)$:

- Growth of savings *vs*. decay of *value* of \$1 [**inflation**]:
  k = interest rate (*e.g.* $k = 0.1$ for 10%); $\lambda = k$

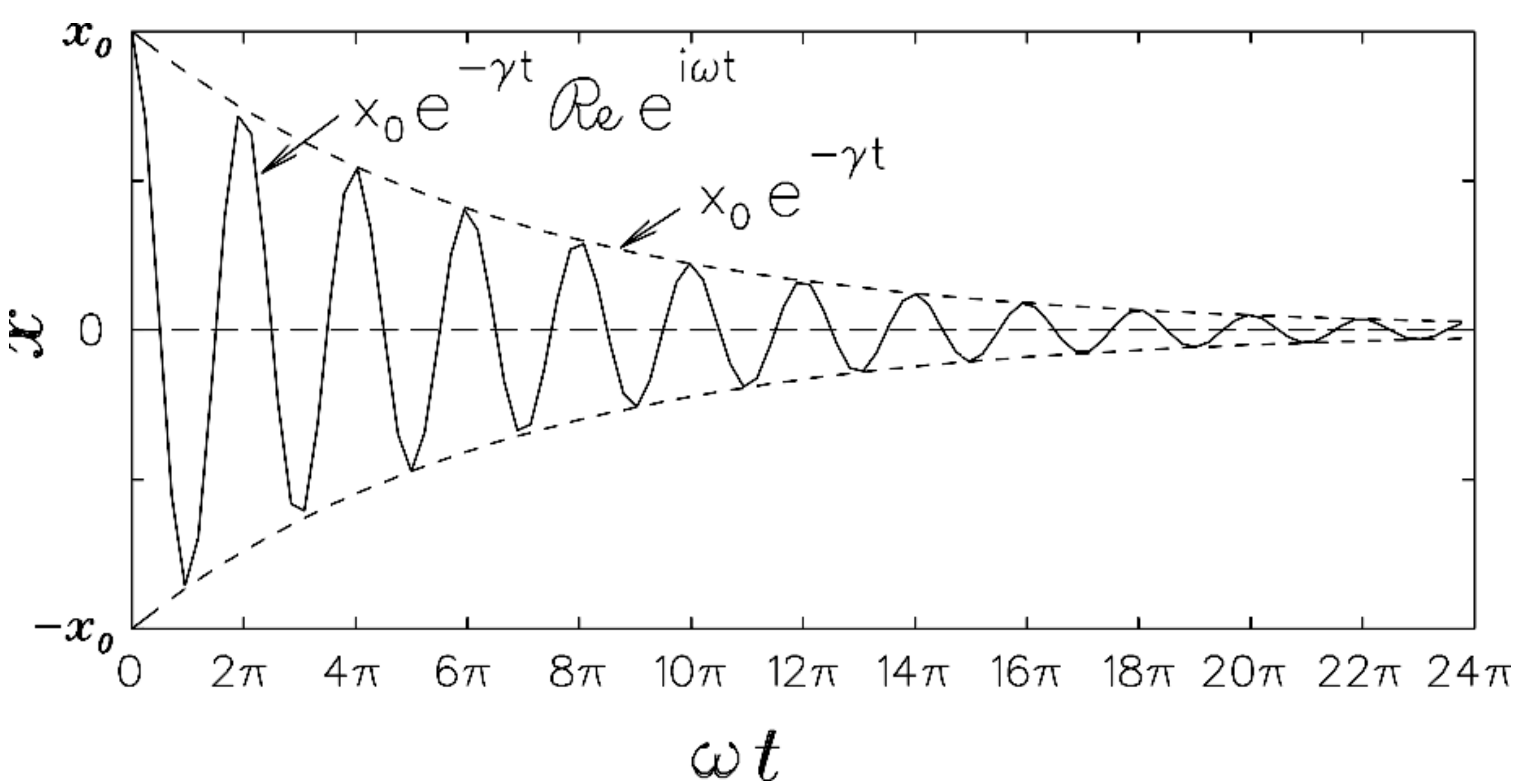
- Propagation of a *pandemic*: $k = R_0 / T_{\text{incub.}}$

- Radioactive decay: $\lambda = 1/\tau = \ell n 2 / T_{1/2}$ $(\ell n 2 = 0.6931478\ldots)$

- Complex exponentials: if $\kappa = -\gamma + i\omega$,
  $e^{\kappa t} = e^{-\gamma t}(\cos \omega t + i \sin \omega t)$ [damped oscillations]

Damped Harmonic Motion:

$$x(t) = x_0\, e^{\kappa t} = x_0\, e^{-\gamma t} \exp(\pm \boldsymbol{i}\, \omega t)$$



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