More Integrals

with

Exponentials & Logarithms

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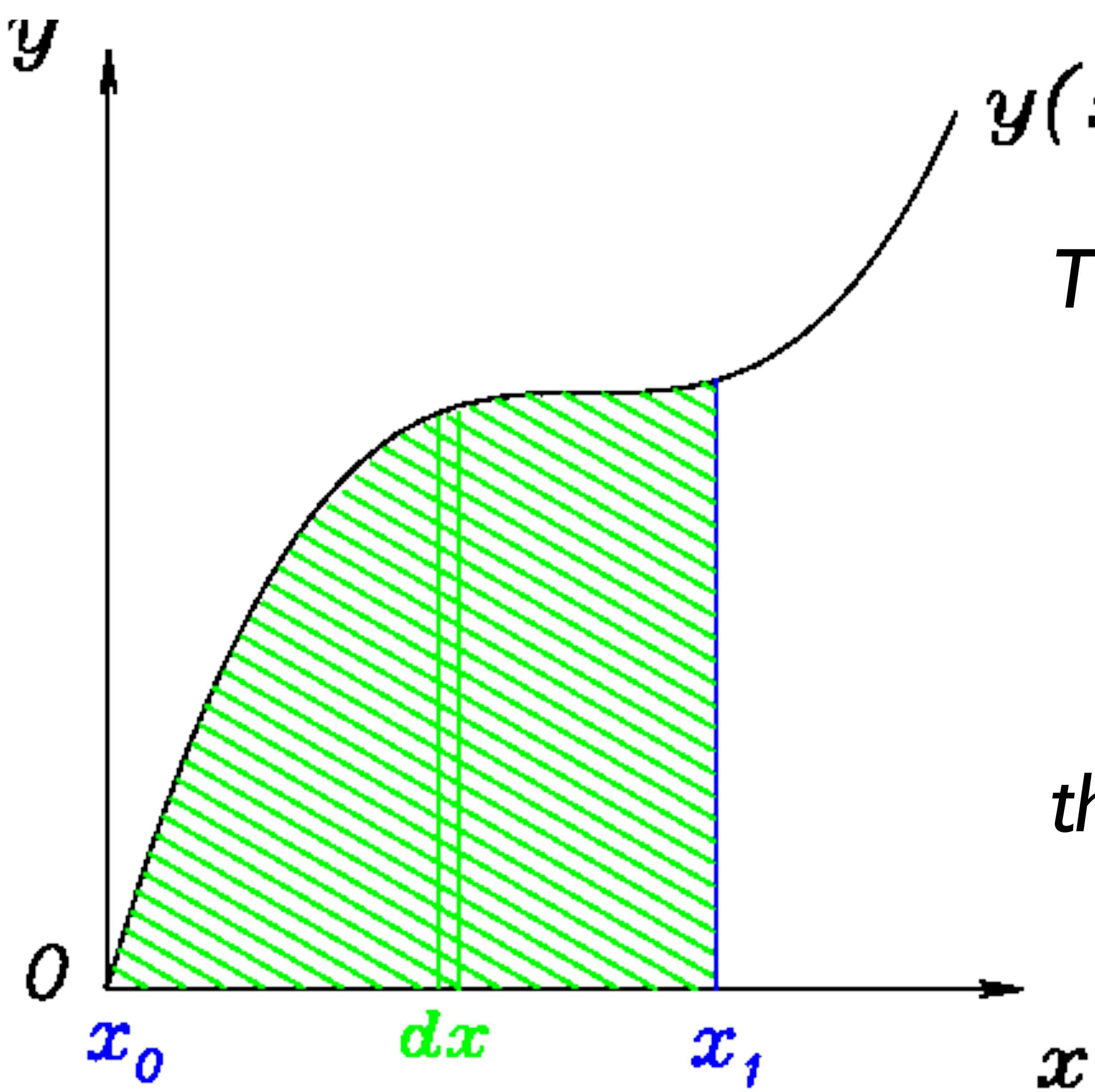
INTEGRALS



The **Definite Integral**

$$\int_{x_0}^{x_1} y(x)dx$$

is *defined* as
the ***area under the curve***
$y(x)$ between x_0 and x_1.

It is described in terms of adding up many vertical "slices" of infinitesimal width dx *and height* $y(x)$.

INTEGRALS

The ***Indefinite Integral*** *(a.k.a.* ***Antiderivative****) of $y(x)$ is better thought of as the function whose derivative is $y(x)$.*

Just ask,

"What Function Has This Derivative?"

If $g(x) = a = df/dx$, what is $f(x)$?

Answer: $f(x) = \int a\, dx = a\, x$ + const.

If $g(x) = 2\, b\, x = df/dx$, what is $f(x)$?

Answer: $f(x) = \int 2\, b\, x\, dx = b\, x^2$ + const.

More Examples

of Indefinite Integrals (Antiderivatives)

If $g(x) = e^x = df/dx$, what is $f(x)$?

Answer: $f(x) = \int e^x \, dx = e^x$ + const.

If $g(x) = 1/e^{kx} \equiv e^{-kx} = df/dx$, what is $f(x)$?

Answer: $f(x) = \int e^{-kx} \, dx = \dfrac{e^{-kx}}{-k}$ + const.

*Wait… How do we **know** that?*

Answer: ***Substitution of Variables****…*

Substitution of Variables

Suppose $u(x)$ is a familiar function and $u'(x)$ is its familiar derivative.[1] Then **if** $y(x)dx$ can be expressed in the form $f[u(x)]u'(x)dx$, we can replace $u'(x)dx$ by du so that[2]

$$\boxed{\int_{x_0}^{x_1} y(x)\, dx = \int_{u(x_0)}^{u(x_1)} f(u)\, du}$$

[1] Remember, $u'(x)$ is Mathematician's notation for du/dx.

[2] Note the use of the *differential* $du \equiv u'(x)\, dx$. It looks almost as if du and dx were regular *quantities* that we could do algebra with at will. We Physicists play fast and loose with differentials, while Real Mathematicians wince the way you might when observing someone riding a bicycle "no hands" down a busy street, blindfolded. (We're not really unable to see where we're going; our blindfolds are just translucent, not opaque. :-)

Substitution of Variables

$$f(x) = \int g[u(x)]\, dx = \int \frac{g(u)\, du}{du/dx}$$

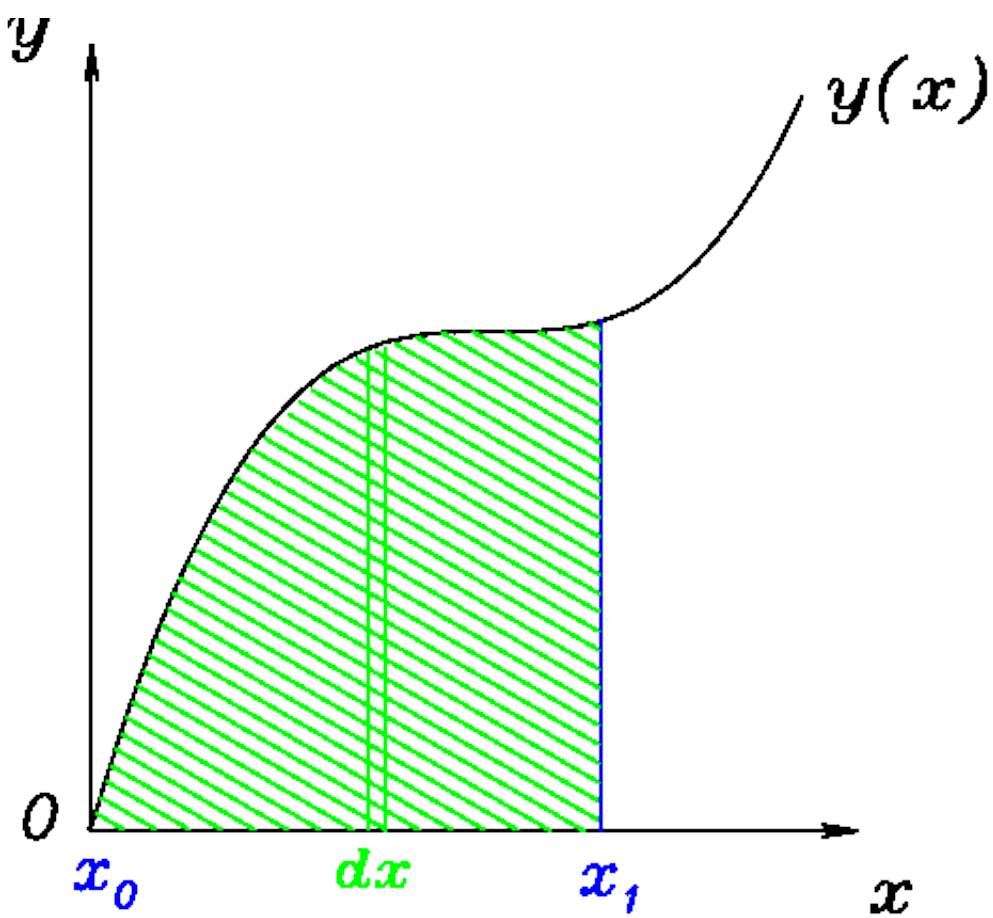
If $g(x) = 1/e^{kx} \equiv e^{-kx} = df/dx$, what is $f(x)$?

Let $u = -kx$ *so that* $du/dx = -k$

$$f(x) = \int e^{u(x)}\, dx = \int \frac{e^{u}\, du}{du/dx} = \frac{e^{u}}{-k} + \text{const.}$$

or $f(x) = \int e^{-kx}\, dx = \dfrac{e^{-kx}}{-k}$ + const. (*QED*)

INTEGRALS



The **Definite Integral**

$$\int_{x_0}^{x_1} y(x)dx$$

is defined as

the ***area under the curve***

$y(x)$ between x_0 and x_1.

But it is equal to the ***difference*** *between the*

Antiderivative $f(x)$ *of $y(x)$ at the endpoints:.*

$$\int_{x_0}^{x_1} y(x)dx = f(x_1) - f(x_0)$$

More Examples

of ***Definite Integrals***

If $y(x) = x^{-1}$, $x_0 = 1$ and $x_1 = 2$, what is $\int_{x_0}^{x_1} y(x)dx$?

Answer: $\ell n(x_1/x_0) = \ell n(2/1) = \ell n(2) = 0.6931478\ldots$

If $y(x) = x\,e^x$, $x_0 = 2$ and $x_1 = 4$, what is $\int_{x_0}^{x_1} y(x)dx$?

Answer: $4e^4 - 2e^2 - (e^4 - e^2) = 3e^4 - e^2 \simeq 156.40539$

Wait… How do we ***know*** *that?*

Answer: ***Integration by Parts***…

Integration by Parts

Sometimes there are two functions of x, $u(x)$ and $v(x)$, with familiar derivatives such that $\int f(x)dx$ can be expressed in the form $\int u\ dv$ where $dv \equiv v'(x)\ dx$. Then

$$\boxed{\int_{x_0}^{x_1} f(x)dx = [u\ v]_{x_0}^{x_1} - \int_{u(x_0)}^{u(x_1)} v\ du}$$

where $[u\ v]_{x_0}^{x_1} \equiv u(x_1)v(x_1) - u(x_0)v(x_0)$.

Integration by Parts

If $y(x) = x\,e^x$, $x_0 = 2$ and $x_1 = 4$, what is $\int_{x_0}^{x_1} y(x)dx$?

Let $u = x$ *and* $v = e^x$ *so that* $du = dx$ *and* $dv = e^x\,dx$

$$\int_{x_0}^{x_1} u\,dv = u(x_1)\,v(x_1) - u(x_0)\,v(x_0) - \int_{x_0}^{x_1} v\,du$$

$$= x_1\,e^{x_1} - x_0\,e^{x_0} - (e^{x_1} - e^{x_0})$$

$$= 4e^4 - 2e^2 -(e^4 - e^2) = 3e^4 - e^2 \simeq 156.40539$$

(*QED*)