Easy Integrals

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1 Antiderivatives

One way to think of an *integral* is as sort of *the opposite of a derivative* — sometimes called (to the dismay of Real Mathematicians) "*antiderivatives*". We can "solve" antiderivatives the same way we "solve" long division problems: by trial and error guessing! Suppose we are given an explicit function $f(x)$ [for example, $f(x) = x^2$] and told that $f(x)$ *is the derivative of* a function $y(x)$ which we would like to know — that is,

$$\text{If} \quad \frac{dy}{dx} = x^2, \quad \text{what is} \quad y(x)\,?$$

Well, we know that

$$\frac{d}{dx}\left[x^3\right] \;=\; 3\,x^2,$$

so we must divide by 3 to get

$$y(x) = \tfrac{1}{3}\,x^3 \;+\; y_0$$

where the constant term y_0 (the value of y when $x = 0$) cannot be determined from the information given — the derivative of any constant is zero, so such an *integral* is always undetermined to within such a *constant of integration*.

But what about the *range* of integration? The procedure described above actually defines the INDEFINITE INTEGRAL,

$$y(x) = \int x^2 dx.$$

What about the DEFINITE INTEGRAL,

$$A = \int_{x_0}^{x_1} x^2 dx.$$

The prescription in that case is to just "*plug in*" the values of x at the upper and lower limits to the expression obtained as the ANTIDERIVATIVE and calculate their *difference*:

$$A = \left(\frac{1}{3}x_1^3 + y_0\right) - \left(\frac{1}{3}x_0^3 + y_0\right) = \frac{1}{3}\left(x_1^3 - x_0^3\right)$$

(Note how the constant of integration cancels out in the definite integral.)

2 Constant times a Function

$$\int a \cdot f(x) dx = a \cdot \int f(x) dx$$

(the integral of a constant times a function is the constant times the integral of the function). This is easy to see in terms of the *area under the curve*: if the curve is raised by a factor of a, the area under it is raised by that same factor.

3 Power of x

The same reasoning that gave us the integral of x^2 can be extended to the integral of x^p,

$$\boxed{\text{If } \; y(x) = x^p, \quad \int y(x) dx = \frac{x^{p+1}}{p+1} + y_0}$$

with one important exception: $p \neq -1$. What does x^{-1} mean? It's the thing you multiply $x^{+1} = x$ by to get $x^0 = 1$. (Any number raised to the zeroth power is 1.) So $x^{-1} = 1/x$ and in fact $x^{-p} = 1/x^p$. What, then, is

$$\int \frac{dx}{x}\;?$$

You can make a plot of the curve $y(x) = 1/x$ and see by inspection that the area under that curve is not zero; so the result we're specifically excluding is

$$\int x^{-1}dx \neq \frac{x^{-1+1}}{-1+1} = \frac{x^0}{0} = \frac{1}{0} \to \infty.$$

So what *is* $\int x^{-1}dx$? As stated in the chapter on Exponentials, it is the NATURAL LOGARITHM of x,

$$\boxed{\int_{x_0}^{x_1} \frac{dx}{x} = \ln x_1 - \ln x_0 = \ln\left(\frac{x_1}{x_0}\right)}$$

4 Exponentials

The derivative of $\exp(kx)$ (where k is a constant) is

$$\frac{d}{dx}e^{kx} = ke^{kx}$$

so the *antiderivative* of $\exp(kx)$ is just

$$\boxed{\int_{x_0}^{x_1} e^{kx}dx = \frac{1}{k}\left[e^{kx_1} - e^{kx_0}\right]}$$

5 Substitution of Variables

Suppose $u(x)$ is a familiar function and $u'(x)$ is its familiar derivative.[1] Then **if** $y(x)dx$ can be expressed in the form $f[u(x)]u'(x)dx$, we can replace $u'(x)dx$ by du so that[2]

$$\boxed{\int_{x_0}^{x_1} y(x)\, dx = \int_{u(x_0)}^{u(x_1)} f(u)\, du}$$

A trivial example is when $u(x) = kx$ (a constant times x). Then we substitute u/k for x and du/k for dx. This is helpful when integrating (for example)[3]

$$\int e^{kx}dx = \frac{1}{k}\int e^u du = \frac{1}{k}e^u = \frac{1}{k}e^{kx}$$

6 Integration by Parts

Sometimes there are two functions of x, $u(x)$ and $v(x)$, with familiar derivatives such that $\int f(x)dx$ can be expressed in the form $\int u\ dv$ where $dv \equiv v'(x)\ dx$. Then

$$\boxed{\int_{x_0}^{x_1} f(x)dx = [u\ v]_{x_0}^{x_1} - \int_{u(x_0)}^{u(x_1)} v\ du}$$

where $[u\ v]_{x_0}^{x_1} \equiv u(x_1)v(x_1) - u(x_0)v(x_0)$.

See if you can use INTEGRATION BY PARTS to find the definite integral

$$\int_0^1 xe^{-kx}dx$$

[1] Remember, $u'(x)$ is Mathematician's notation for du/dx.

[2] Note the use of the *differential* $du \equiv u'(x)\ dx$. It looks almost as if du and dx were regular *quantities* that we could do algebra with at will. We Physicists play fast and loose with differentials, while Real Mathematicians wince the way you might when observing someone riding a bicycle "no hands" down a busy street, blindfolded. (We're not really unable to see where we're going; our blindfolds are just translucent, not opaque. :-)

[3] Remember, $\exp(u)$ is its own derivative and therefore also its own integral!