Simple Harmonic Motion

Many types of time-dependence are ***oscillatory***,

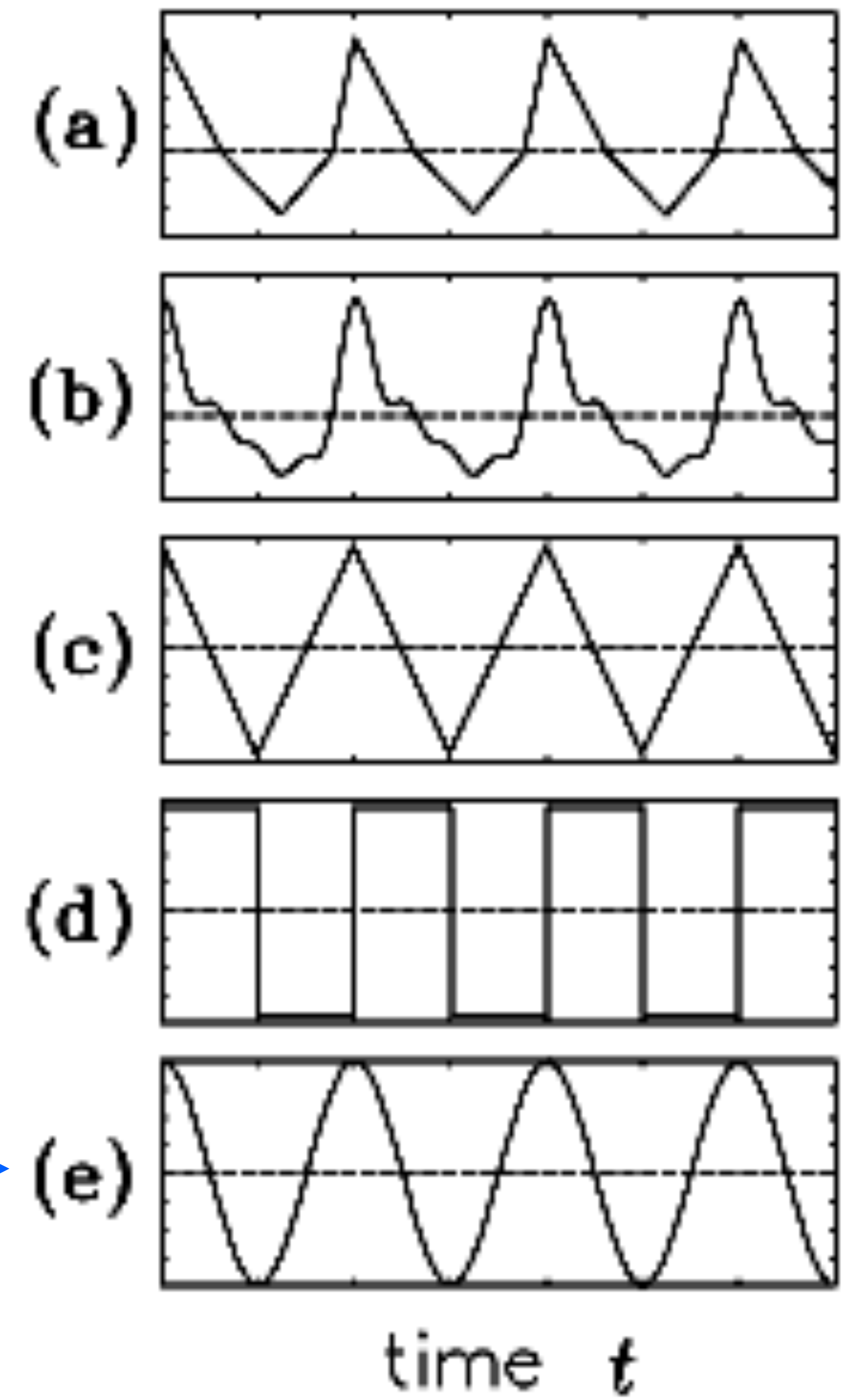
many are ***periodic***,

but only one type is "***harmonic***",

namely,

***sinusoidal* motion.** →

What are its properties?

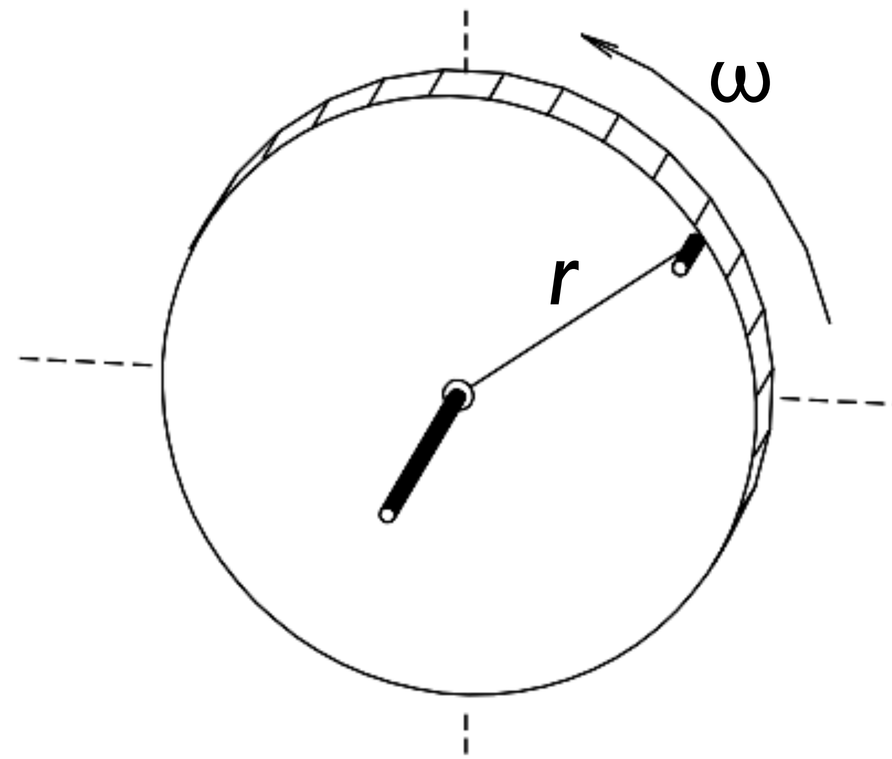


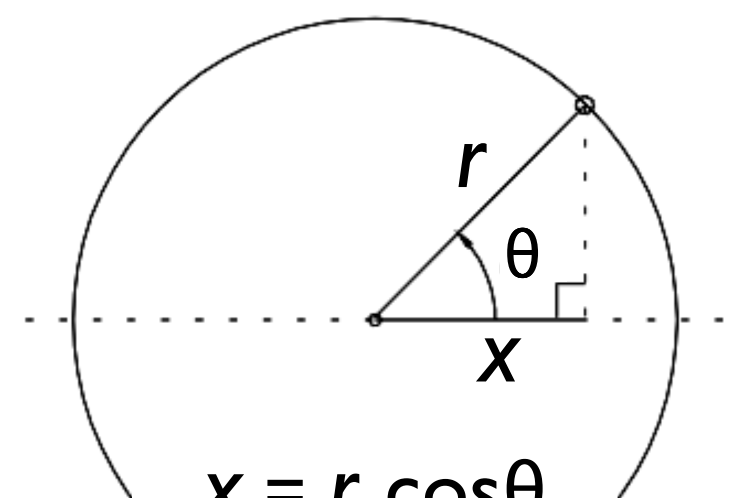
Projecting the Wheel

Picture a wheel spinning at constant angular velocity ω.

Now picture the motion of the ***shadow*** of a ***pin*** on the ***rim*** of the wheel (at high noon on the Equator).

This is called (reasonably) the ***projected*** *motion* of the pin.





$$x = r\cos\theta$$

$$\theta = \omega t$$

A Little Mathematics (SWOP):

Small Angles:

$$\text{For} \quad \theta \ll 1, \quad \cos(\theta) \approx 1 - \tfrac{1}{2}\theta^2$$
$$\text{and} \quad \sin(\theta) \approx \theta.$$

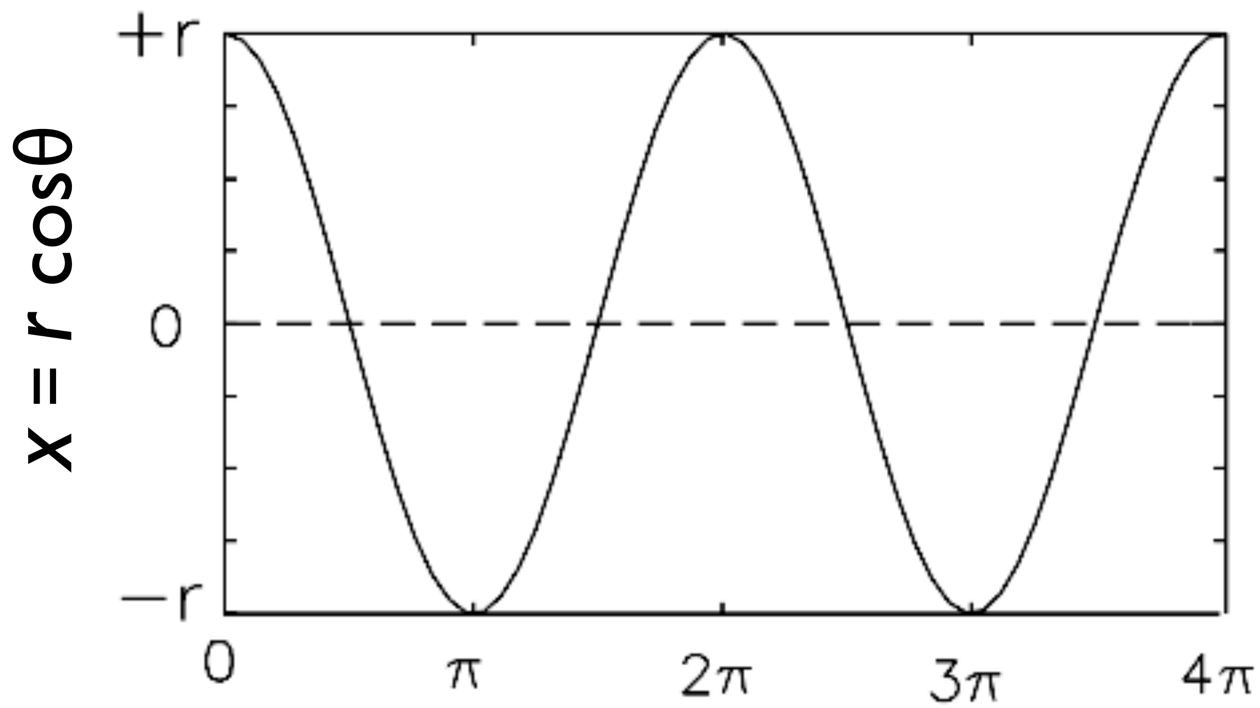
Taylor Series Expansions for Exponential & Sinusoidal Functions:

$$\exp(z) = 1 + z + \tfrac{1}{2}z^2 + \tfrac{1}{3!}z^3 + \tfrac{1}{4!}z^4 + \cdots$$

$$\cos(z) = 1 - \tfrac{1}{2}z^2 + \tfrac{1}{4!}z^4 - \cdots$$

$$\sin(z) = z - \tfrac{1}{3!}z^3 + \cdots$$

Derivatives of the Cosine Function



$\theta = \omega t + \varphi$

$x = r \cos(\omega t + \varphi)$

Note:

$d^2x/dt^2 = -\omega^2 x$

$v \equiv dx/dt = -\omega r \sin(\omega t + \varphi)$

$a \equiv d^2x/dt^2 = -\omega^2 r \cos(\omega t + \varphi)$

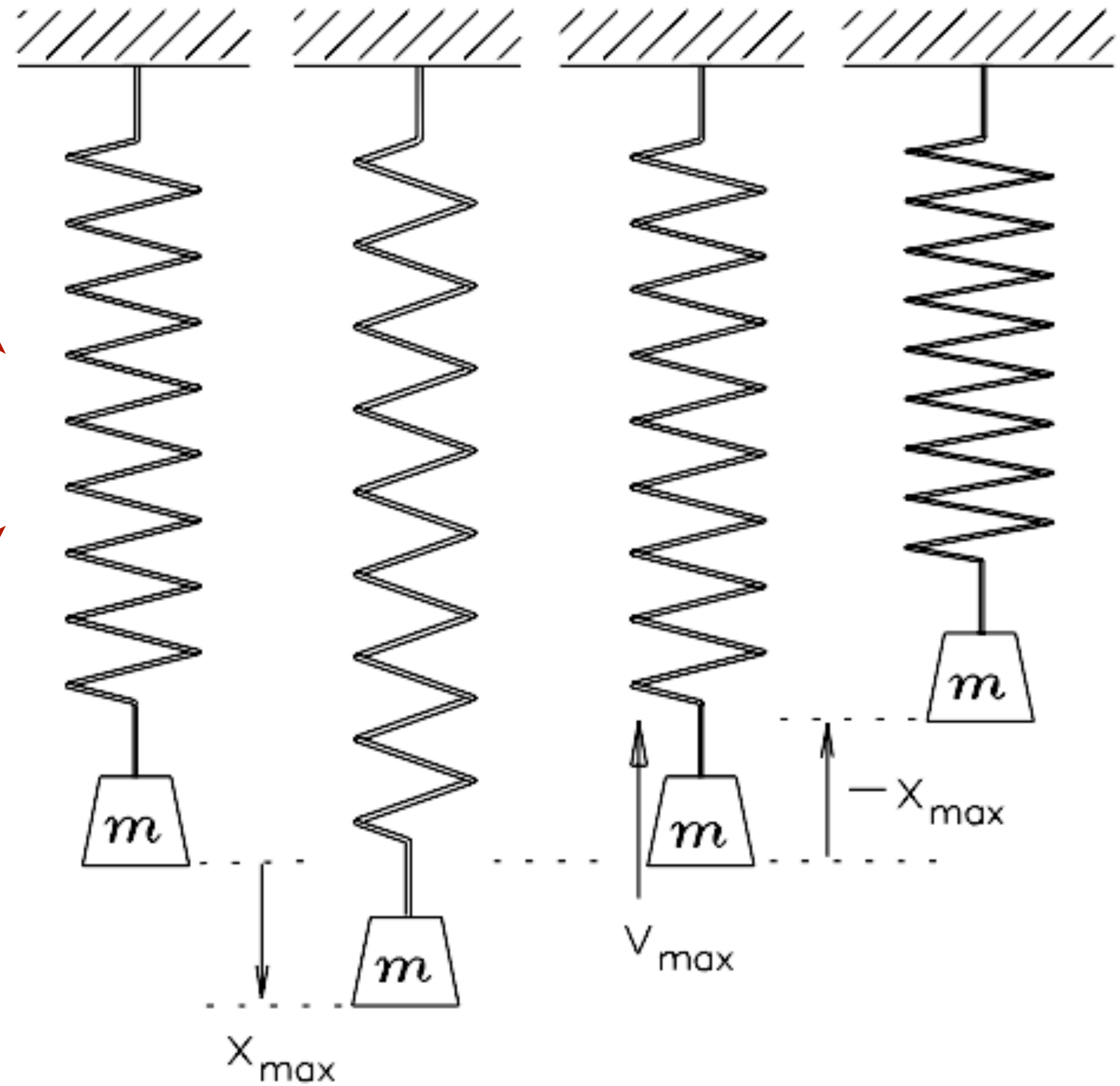
The Spring Pendulum

$F = ma$

Hooke's Law

$F = -kx$

$a = -(k/m)x$



So $\quad a \equiv d^2x/dt^2 = -\omega^2 x \quad$ if $\quad \omega^2 = k/m \quad$ or $\quad \omega = \sqrt{k/m}$

Our old friend, the Exponential:

Remember $dx/dt = -\kappa x \Leftrightarrow x(t) = x_0 \exp(-\kappa t)$?

The second derivative would be $d^2x/dt^2 = \kappa^2 x$, right?

Well, now we have a new equation $d^2x/dt^2 = -\omega^2 x$, which means the *exponential* function would be a solution if only we could have $\kappa^2 = -\omega^2$.

Of course this is impossible. No real number is negative when squared.

But what if there *were* such a number? *Use your imagination!* $i = \sqrt{-1}$

Then we could solve our differential equation in one step:

$$x(t) = x_0 \exp(i\,\omega\,t)$$

— but what does this ***mean***?

Complex Exponentials

Taylor Series Expansions for Exponential & Sinusoidal Functions:

Recall

$$\exp(z) = 1 + z + \tfrac{1}{2}z^2 + \tfrac{1}{3!}z^3 + \tfrac{1}{4!}z^4 + \cdots$$

$$\cos(z) = 1 - \tfrac{1}{2}z^2 + \tfrac{1}{4!}z^4 - \cdots$$

$$\sin(z) = z - \tfrac{1}{3!}z^3 + \cdots$$

What if $z = i\theta$?

$$\exp(i\theta) = 1 + i\theta - \tfrac{1}{2!}\theta^2 - \tfrac{1}{3!}i\theta^3 + \tfrac{1}{4!}\theta^4 + \cdots$$

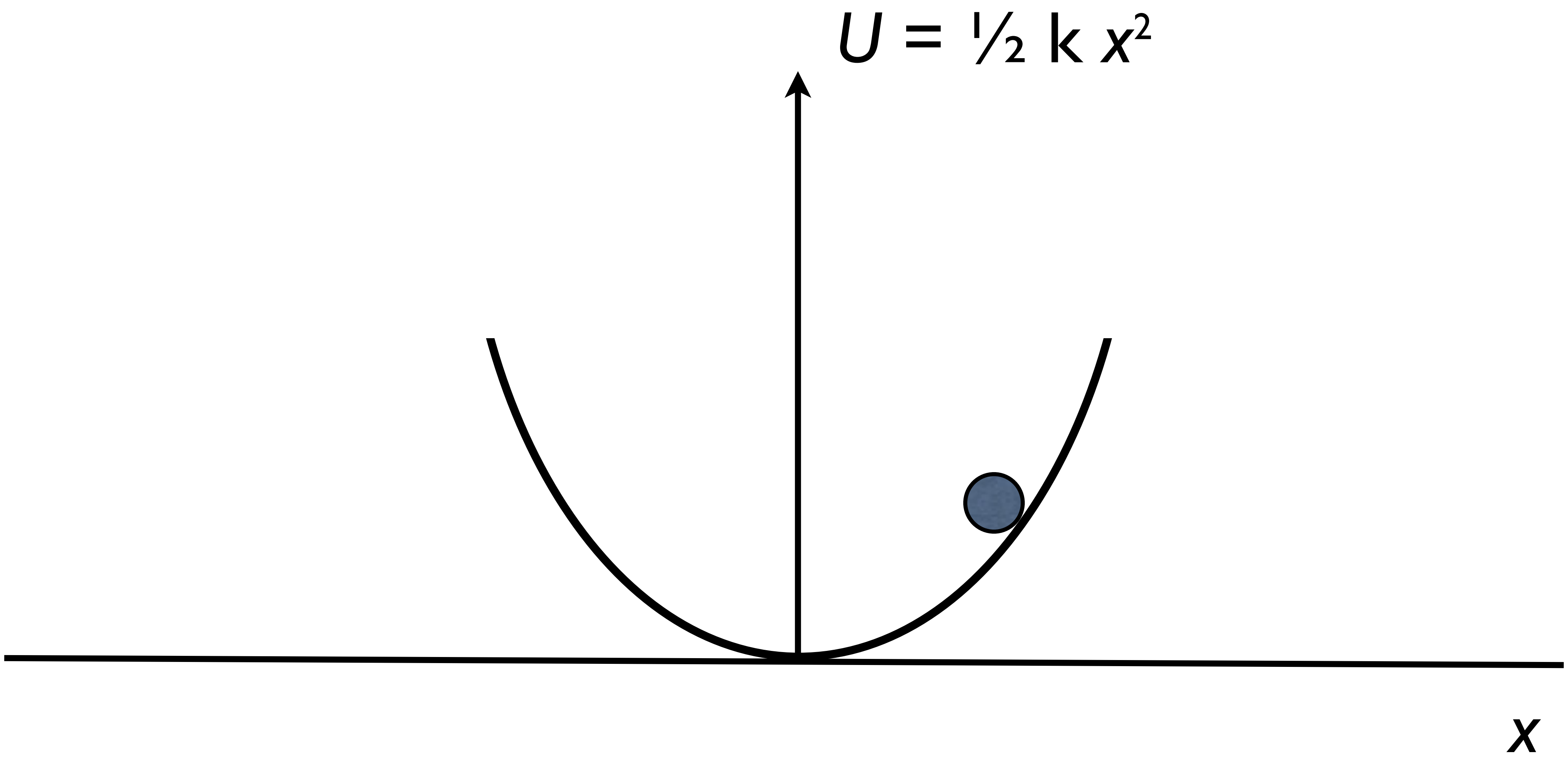
$$\cos(\theta) = 1 - \tfrac{1}{2!}\theta^2 + \tfrac{1}{4!}\theta^4 + \cdots$$

$$i\sin(\theta) = i\theta - \tfrac{1}{3!}i\theta^3 + \cdots$$

This means

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Quadratic Potential Minimum



$F = -dU/dx = -k\,x$

Simple Harmonic Motion

Linear Restoring Force
(Hooke's Law)

$F = -k\,x$

$\Updownarrow$

Quadratic Potential Minimum

$U = ½\,k\,x^2$

plus

Inertial Factor m

$\Leftrightarrow$

SHM

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

$\Updownarrow$

$$x = x_0 \cos(\omega t + \varphi)$$

$$v \equiv \frac{dx}{dt} = -\omega\, x_0 \sin(\omega t + \varphi)$$

$$a \equiv \frac{d^2x}{dt^2} = -\omega^2\, x_0 \cos(\omega t + \varphi)$$

$$\omega^2 = k/m$$

Damped Harmonic Motion:

Viscous damping:

$$d^2x/dt^2 = -\kappa\, dx/dt \quad \Leftrightarrow \quad v(t) = v_0 \exp(-\kappa t)$$

With a linear restoring force ***and*** viscous damping, the equation is

$$d^2x/dt^2 = -\kappa\, dx/dt - \omega^2 x$$

which still might be satisfied by $x(t) = x_0 \exp(Qt)$ with some Q.

Let's try! Plug this $x(t)$ back into the equation, giving

$$Q^2 x = -\kappa Q x - \omega^2 x \qquad \text{or} \qquad Q^2 + \kappa Q + \omega^2 = 0$$

which has the solution $2Q = -\kappa \pm \sqrt{\kappa^2 - 4\omega^2}$

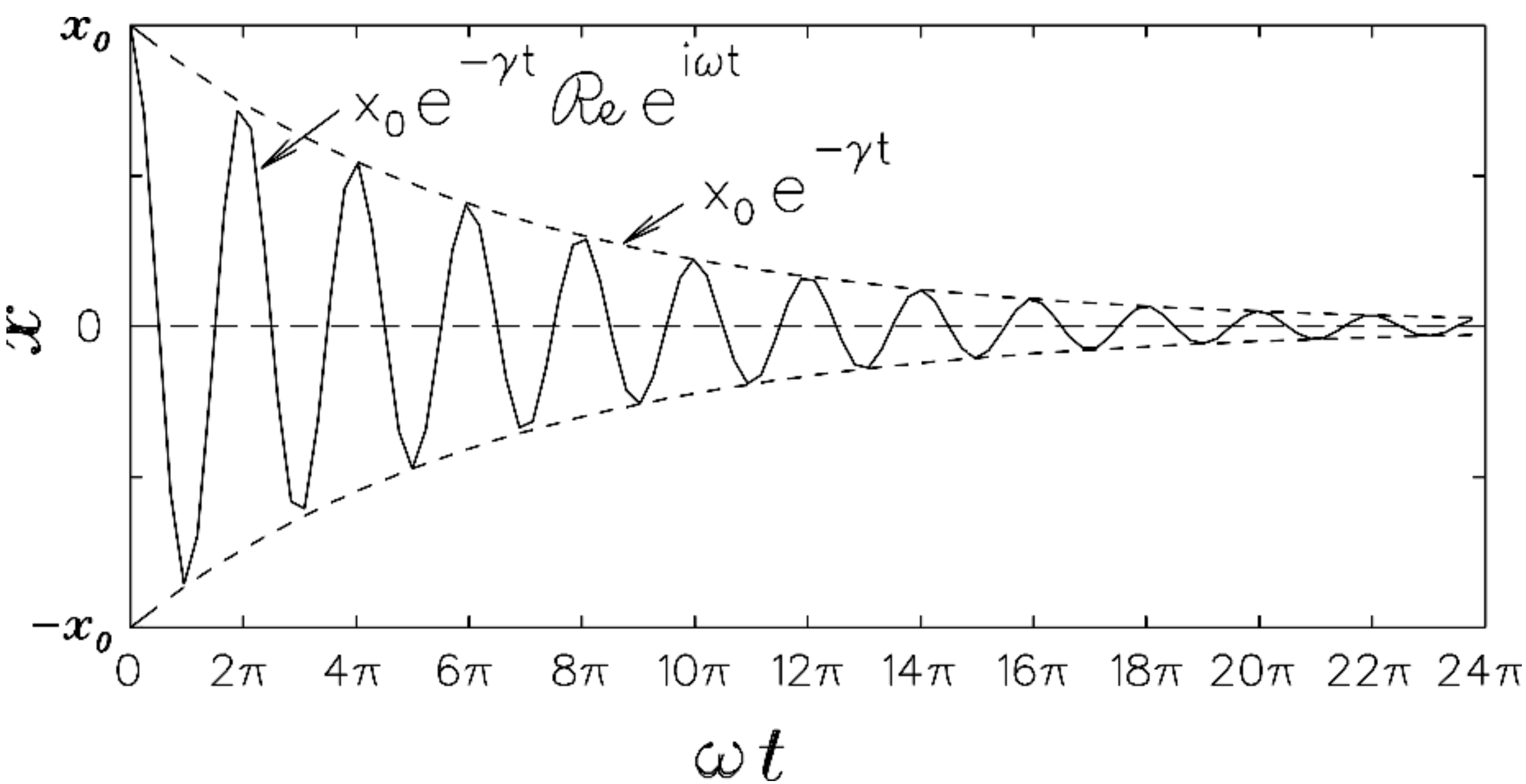
Again, *what does this* ***mean***?

Damped Harmonic Motion:

If $\quad 2Q = -\kappa \pm \sqrt{\kappa^2 - 4\omega^2}$ $\quad$ then

$Q = -\tfrac{1}{2}\kappa \pm i\,\omega\sqrt{1 - \tfrac{1}{4}\kappa^2/\omega^2}$, $\quad$ so

$x(t) = x_0\, e^{-\frac{1}{2}\kappa t} \exp(\pm\, i\, \omega'\, t)$ $\quad$ where $\quad \omega' = \omega\sqrt{1 - \tfrac{1}{4}\kappa^2/\omega^2}$



The End

for now