Vectors

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1. If you walk 30 m directly across a football field and then 40 m lengthwise (toward the goal-posts), how much further have you walked than if you just walked directly from your starting point to your destination, "as the crow flies"? ANSWER: Call the direction across the field x and the lengthwise direction y. These directions are perpendicular (orthogonal) so the total distance d is given by the PYTHAGOREAN THEOREM: $d^2 = x^2 + y^2$ or, in this case, $d = \sqrt{30^2 + 40^2} = 50$ m, which is $\boxed{30 + 40 - 50 = 20 \text{ m}}$ extra distance.

2. If you're relaxing on a beach, how far away is an airplane that is 10 km North and 10 km East, flying at an altitude of 10,000 m? ANSWER: The vector from you to the airplane is $\vec{\mathbf{r}} = (\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}})$ times 10 km, so the total distance is $\sqrt{1^2 + 1^2 + 1^2} \times 10$ km or $\sqrt{3} \times 10$ km or about $\boxed{17.32 \text{ km}}$.

3. Find the SCALAR PRODUCT of these two vectors: $\vec{\mathbf{A}} = \hat{\imath} + \hat{\jmath} + \hat{k}$ and $\vec{\mathbf{B}} = \hat{\imath} - \hat{\jmath} - \hat{k}$ ANSWER: $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_x B_x + A_y B_y + A_z B_z = (+1)(+1) + (+1)(-1) + (+1)(-1) = 1 - 1 - 1$ or $\boxed{\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = -1}$. The minus sign means that $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ are predominantly in *opposite directions*.

4. What is the UNIT VECTOR in the direction of the vector $\vec{\mathbf{A}} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$? ANSWER: To get the UNIT VECTOR $\hat{\mathbf{A}}$, we divide $\vec{\mathbf{A}}$ by its own magnitude $A = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14} \approx 3.7416574$: $\boxed{\hat{\mathbf{A}} = \dfrac{\hat{\imath} + 2\hat{\jmath} + 3\hat{k}}{\sqrt{14}}}$ or $\hat{\mathbf{A}} \approx 0.26726124\hat{\imath} + 0.53452248\hat{\jmath} + 0.80178372\hat{k}$. (In a marked assignment, preference would be given to the *exact* answer in terms of $\sqrt{14}$.)

5. What is the VECTOR PRODUCT of these two vectors: $\vec{\mathbf{A}} = \hat{\imath} + \hat{\jmath} + \hat{k}$ and $\vec{\mathbf{B}} = \hat{\imath} - \hat{\jmath} - \hat{k}$? ANSWER: Recall the definition of the vector (or "cross") product:

$$\begin{aligned} \vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}} &= \hat{\imath}(A_y B_z - A_z B_y) \\ &+ \hat{\jmath}(A_z B_x - A_x B_z) \\ &+ \hat{k}(A_x B_y - A_y B_x) \,. \end{aligned} \tag{1}$$

so in this case

$$\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}} = \hat{\imath}\,[(1)(-1) - (1)(-1)] + \hat{\jmath}\,[(1)(1) - (1)(-1)] + \hat{k}\,[(1)(-1) - (1)(1)]$$

or $\boxed{\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}} = 2\hat{\jmath} - 2\hat{k}}$.