THE UNIVERSITY OF BRITISH COLUMBIA

Physics 108 Assignment # 1 SOLUTIONS:

THERMAL PHYSICS

Wed. 05 Jan. 2005 — finish by Wed. 12 Jan.

1. "STAT - EC": Consider the following simplified model of a sort of *stock market*: A given stock $\mathcal{S}$ has a total of N shares on the market for a fixed price ε. At a given time, n of these shares are bought and the remaining $N-n$ are unwanted. Thus the net investment in $\mathcal{S}$ is $U = n\varepsilon$. [Here ε and U are measured in *monetary* units, say dollars; I have used the same notation as for *energy* for reasons that will soon become evident.] To keep things simple, we shall assume that the *price* ε of a given stock *does not change*. Further, let's make the outrageous assumption that the stock market as a whole is *a priori equally likely to be found in any one of the fully specified states accessible to it* — *i.e.* that a given amount of capital is equally likely to be distributed amongst the various stocks in any of the possible ways that give the same total.[1]

(*a*) Invent a general definition for an economic analogue of *temperature* τ [measured in monetary units] that has the desired predictive power: that (given our starting assumptions) capital will tend to flow spontaneously from stocks with *higher* τ into others with *lower* τ and will stop flowing between two stocks only when they are in "economic equilibrium" — *i.e.* when they have the same "economic temperature" τ.

ANSWER: This can be copied right off the handout; I just wanted you to give some thought to the implications. If $\Omega(U) \equiv \Omega(n,N)$ is the number of different ways the total capital investment $U = n\varepsilon$ in stock $\mathcal{S}$ can be redistributed among the N shares available ($N > n$) of that stock, then the ENTROPY of stock $\mathcal{S}$ is $\sigma = \ln\Omega$. We even have a formula for $\Omega(n,N)$ given the above specifications, namely the *binomial distribution*, but this part of the question is completely general and you need not work out the actual specific result, just define the inverse economic temperature as $\boxed{\dfrac{1}{\tau} \equiv \dfrac{\partial\sigma}{\partial U},}$ in terms of which the above-mentioned predictive power comes automatically!

Just for fun, I will go ahead and derive the explicit U-dependence of τ for the binomial distribution: $\Omega(n,N) = \dfrac{N!}{n!(N-n)!}$ or $\Omega(U) = \dfrac{(2\hat{U}/\varepsilon)!}{(U/\varepsilon)![(2\hat{U}-U)/\varepsilon]!}$ where $\hat{U} \equiv \frac{1}{2}N\varepsilon$ is the energy giving the largest possible Ω. For large N this can be approximated by a gaussian distribution: $\Omega(U) \approx \Omega(\hat{U})\exp\left[-2(U-\hat{U})^2/N\varepsilon^2\right]$, giving $\sigma(U) \equiv \ln\Omega(U) = c - (U-\hat{U})^2/\hat{U}\varepsilon$. Thus $\dfrac{1}{\tau} = \dfrac{\partial\sigma}{\partial U} = -2\dfrac{(U-\hat{U})}{\hat{U}\varepsilon} = 2\dfrac{1-U/\hat{U}}{\varepsilon}$ or $\tau = \dfrac{\varepsilon}{2(1-U/\hat{U})}$. Note that τ has the same dimensions as ε, as expected. For your own amusement, consider how the economic temperature of stock $\mathcal{S}$ behaves when half or more of its available shares are sold!

(*b*) Now assume that the entire market is in "economic equilibrium" and is so much larger than any of its parts that we may treat it as a "capital reservoir" $\mathcal{R}$ at an "economic temperature" of $\tau =$\$100. Consider one share of one stock, valued at $\varepsilon_1 =$\$200: What is the probability that it will be bought at any given time?

ANSWER: Again, this is right out of the handout: we have now the analogue of the CANONICAL ENSEMBLE, in which the probability of one fully specified state of a "microsystem" (in this case a single share of a given stock) follows the BOLTZMANN DISTRIBUTION: $\mathcal{P}_\alpha \propto e^{-\varepsilon_\alpha/\tau}$. In order to convert the $\propto$ symbol into an $=$ sign, we need to *normalize* the probability distribution: all the possibilities together must add up to a probability of unity. In this case there are only two possibilities: either the stock is *bought* (investment ε, probability $\mathcal{P}_b = Ce^{-\varepsilon/\tau}$) or it is *not bought* (investment 0, probability $\mathcal{P}_{nb} = Ce^{-0/\tau} = C$), where the constant of proportionality C must be adjusted to make the total come out to $1 = \mathcal{P}_b + \mathcal{P}_{nb} = C(1+e^{-\varepsilon/\tau})$. Thus $C = 1/(1+e^{-\varepsilon/\tau})$ and we have $\mathcal{P}_b = \dfrac{e^{-\varepsilon/\tau}}{1+e^{-\varepsilon/\tau}} = \dfrac{1}{e^{+\varepsilon/\tau}+1}$, in this case $\boxed{\mathcal{P}_b = \dfrac{1}{e^{\$200/\$100}+1} = \dfrac{1}{e^2+1} = 0.1192.}$

(*c*) Assuming that $\mathcal{R}$ is also huge compared to the entire offering of $N_1 = 1000$ shares of stock $\mathcal{S}_1$ valued at $\varepsilon_1 =$\$200, what is the expected total investment U_1 in $\mathcal{S}_1$ when $\tau =$\$100?

ANSWER: If the probability of *any one* share of $\mathcal{S}$ being bought is $\mathcal{P}_b = 1/(e^2+1)$, then the expected average total investment U in $\mathcal{S}$ is $\langle U\rangle = N\mathcal{P}_b\varepsilon = 1000 \times \dfrac{1}{e^2+1} \times \$200$ or $\boxed{\langle U\rangle = \$23,841.}$

(*d*) If the economic temperature drops to $\tau =$\$50, which stock will be likely to have the most capital U invested in it, $\mathcal{S}_1$ with $N_1 = 1000$ shares at $\varepsilon_1 =$\$200 per share or $\mathcal{S}_2$ with $N_2 = 1000$ shares at $\varepsilon_2 =$\$100 per share?

ANSWER: As in the previous part, $\langle U_1\rangle = N_1 \times \dfrac{1}{e^{200/50}+1} \times \varepsilon_1 = 1000 \times \dfrac{1}{e^4+1} \times \$200 = \$3,597$ whereas $\langle U_2\rangle = N_2 \times \dfrac{1}{e^{100/50}+1} \times \varepsilon_2 = 1000 \times \dfrac{1}{e^2+1} \times \$100 = \$11,920$. Thus the stock selling for the lower price will tend to attract the larger *net investment* for the same number of shares, even though each share is sold for less, as long as both are *expensive* — *i.e.* selling for well above the market's economic temperature τ. (If $\varepsilon < \tau$ then almost half the shares of the stock will be sold and the stock with the larger price per share will attract a larger net investment. Notice how important it is to be aware of

[1] This is not consistent with current economic theory, which focusses on "rational agents." Here we assume totally mindless, random investment decisions.

the market's "temperature.") Of course, all this is founded on several assumptions — totally mindless investment decisions and the consequent lack of "feedback" affecting the prices of stocks — which cannot possibly describe the behaviour of a real stock market ... can they? Note also that because there is a limit on how much money can be spent on a given stock (namely when all shares are sold, a state of zero entropy) we can have all the peculiar effects of *negative "economic temperature"* — but for $\tau > 0$ the maximum fraction of shares sold will be 50%.

2. MARS-EQUIVALENT ATMOSPHERIC PRESSURE: The composition of Mars' atmosphere is nominally 95.3% CO_2, 2.7% N_2, 1.6% Ar, 0.15% O_2 and 0.03% H_2O. Mean atmospheric pressure at the surface of Mars is 1-9 millibar, depending on altitude; the average is about 7 mb, compared to 1000 mb at sea level on Earth. *At what altitude* here on Earth would the atmospheric pressure be the same as that at the surface of Mars? (Assume an *isothermal* Earth atmosphere at 300 K. Are any other assumptions needed?)

ANSWER: As we have seen, molecules of mass M in an isothermal atmosphere have a relative probability $\mathcal{P}(h) \propto \exp(-Mgh/kT)$ of being found at altitude h. (Here k is Boltzmann's constant.) Thus the ratio of the atmospheric pressure at height h to that at zero height (*i.e.* sea level) is just $\exp(-Mgh/kT)$, so to drop the pressure by a ratio $7/1000 = 0.007$ we must go to a height h where $\exp(-Mgh/kT) = 0.007$ or $Mgh/kT = -\ln 0.007 = 4.96$ or $h = 4.96kT/Mg$. We know $k = 1.38 \times 10^{-23}$ J/K so $kT = 4.142 \times 10^{-21}$ J, and $g = 9.81$ m/s^2, but now we do have to make an additional assumption: namely, that our atmosphere is "mostly nitrogen" where in fact it is about 79% N_2, 20% O_2 and 1% other gases. The partial pressure of heavier gases (mainly oxygen) will drop off faster with altitude, giving a net pressure that does not have a perfectly exponential h-dependence; but the isothermal assumption is much worse, so we can use the M of N_2 (about $28 \times 1.67 \times 10^{-27}$ kg) to get $h = 4.96 \times 4.142 \times 10^{-21}/28 \times 1.67 \times 10^{-27} \times 9.81 = 44685$ m. That is $\boxed{h = 44.7 \text{ km}}$.

3. ORTHO- *vs.* PARA-HYDROGEN: Molecular hydrogen, H_2, consisting of two protons bound together with two electrons, can form in either the "singlet" state called *parahydrogen*, in which the total spin (intrinsic angular momentum) of the molecule is zero, or in any one of three "triplet" states of *orthohydrogen*, in which the proton spins combine to make a total spin of $1\hbar$ (the fundamental unit of angular momentum). For this problem, all you need to know is that the three triplet states are *degenerate* — *i.e.* they all have the same *energy* relative to the singlet state, namely $\varepsilon_3 = 2.375 \times 10^{-21}$ J. (The energy ε_1 of the singlet state can be taken to be zero, for reference.) Assume that the spin degrees of freedom of the H_2 molecules are unaffected by, but are in thermal equilibrium with, all their other degrees of freedom (like translational, rotational or vibrational). In this case, what *fraction* f_3 of H_2 molecules will be found (on average) in *ortho* states

(*a*) at room temperature (300 K)?

ANSWER: As in the earlier example, the probability of a state $|\alpha\rangle$ of energy ε_α being occupied in thermal equilibrium at temperature τ is given by the Boltzmann distribution, $\mathcal{P}_\alpha = Ce^{-\varepsilon_\alpha/\tau}$, where the constant of proportionality C is to be determined by *normalization*.[2] There are three states of energy ε_3 and one of zero energy, so for normalization we must have $1 = C + 3Ce^{-\varepsilon_3/\tau}$ or $C = \dfrac{1}{1 + 3e^{-\varepsilon_3/\tau}}$. The *fraction* of H_2 molecules in all three triplet states combined is the sum of the three (equal) probabilities of any given molecule being in any of these three degenerate states: $f_3 = 3Ce^{-\varepsilon_3/\tau} = \dfrac{3e^{-\varepsilon_3/\tau}}{1 + 3e^{-\varepsilon_3/\tau}} = \dfrac{3}{e^{\varepsilon_3/\tau} + 3}$. For $\tau = 300k_B = 4.142 \times 10^{-21}$ J, $\varepsilon_3/\tau = 0.5734$ and $e^{\varepsilon_3/\tau} = 1.774$, giving $\boxed{f_3 = \dfrac{3}{1.774 + 3} = 0.628}$

(*b*) at the boiling point of liquid nitrogen at atmospheric pressure (77 K)?

ANSWER: The same formula applies here, with $\tau = 77k_B = 1.063 \times 10^{-21}$ J, $\varepsilon_3/\tau = 2.234$ and $e^{\varepsilon_3/\tau} = 9.337$, giving $\boxed{f_3 = \dfrac{3}{9.337 + 3} = 0.243}$

(*c*) at the freezing point of molecular hydrogen at atmospheric pressure (14 K)?

ANSWER: ...and again here, with $\tau = 14k_B = 1.933 \times 10^{-22}$ J, $\varepsilon_3/\tau = 12.29$ and $e^{\varepsilon_3/\tau} = 2.168 \times 10^5$, giving $\boxed{f_3 = \dfrac{3}{2.168 \times 10^5 + 3} = 1.384 \times 10^{-5}}$

[2] In this case, of course, ε is an actual *energy* and τ is a normal *temperature* measured in regular energy units. (Back to the "real world" of thermal physics.)