THE UNIVERSITY OF BRITISH COLUMBIA

Physics 108 Assignment # 2 SOLUTIONS:
KINETIC THEORY OF GASES

Wed. 12 Jan. 2005 — finish by Wed. 19 Jan.

1. Quantum Tension in a String

A single electron is confined to a single-walled carbon nanotube (SWNT) of length $L = 1$ μm but is free to move up and down the length of the SWNT. (Since the SWNT is only about 1.2 nm in diameter, you may think of it as a long string). If this system is cooled to nearly 0 K so that the electron is in its lowest possible energy state (the "ground state"), what is the *tension* in the SWNT "string" due to the electron's confinement?

Hint: Use de Broglie's hypothesis ($\lambda = h/p$) and think in terms of *standing waves*. Then use a classical picture of a particle of momentum $p = mv$ bouncing back and forth off the ends of the string. . . .

ANSWER: The longest wavelength (ground) state of a particle in a one-dimensional (1D) "box" has $\lambda = 2L$ and therefore (by de Broglie's hypothesis) momentum $p = h/2L$. For a "1D box" this large the electron will be nonrelativistic (you can check this), so we can set $p = mv$ and thus $v = 2L/T = p/m = h/2Lm$ where T is the time for the electron to make a round trip to the other end and back. Thus the electron transfers momentum $2p = h/L$ (by reversing its direction) to one end of the "string" every $T = 4L^2m/h$, generating an average force $F = 2p/T = h^2/4mL^3$. Then we just plug in $L = 10^{-6}$ m, $m = 9.11 \times 10^{-31}$ kg and $h = 6.63 \times 10^{-34}$ J-s to get $F = \frac{6.63 \times 6.63 \times 10^{-68}}{4 \times 9.11 \times 10^{-31} \times 10^{-18}}$ or $\boxed{F = 1.206 \times 10^{-19} \text{ N}}$. Since each end of the SWNT is being pushed "out" by this force, it is the same as the tension in the SWNT.[1]

2. One-Dimensional Ideal Gas

Making use of the EQUIPARTITION THEOREM, derive an equation analogous to the familiar 3D IDEAL GAS LAW ($pV = N\tau$) for an ideal gas confined to a **one**-dimensional "box" of length L. (Some examples would be N electrons moving freely along a single DNA molecule, a *trans*-polyacetylene chain, a SWNT or a "nanowire" made from GaAs/AlGaAs structures.)

ANSWER: The arguments developed for the time-averaged *force* exerted on one wall of a cubical 3D container apply equally well for the force acting on the boundary of a 1D "box" — namely, for a *single* particle bouncing back and forth, $\langle F_1 \rangle = m\langle v_x^2 \rangle / L$, where L is the length of the "box". In this case there is only one direction of motion (degree of freedom) so we can drop the $_x$ subscript on v_x. The EQUIPARTITION THEOREM says that the *thermal* average of the kinetic energy $E_1 = \frac{1}{2}mv^2$ associated with this translational degree of freedom is $\langle E_1 \rangle \equiv \frac{1}{2}m\langle v^2 \rangle = \frac{1}{2}\tau$, giving $m\langle v^2 \rangle = \tau$. If we substitute this back into the formula for $\langle F_1 \rangle$ we get $\langle F_1 \rangle = \tau/L$. For N particles all doing this at the same time, we just multiply by N, giving $\langle F \rangle = N\tau/L$ or (ignoring the fact that the actual force F at any instant fluctuates minutely about the average force $\langle F \rangle$) $\boxed{FL = N\tau}$. This is the 1D equivalent of the IDEAL GAS LAW. Note that the only difference between this derivation and the one for a 3D box is that here we didn't have to introduce the notion of pressure as the force per unit area. This is simpler!

[1] This may not sound like much, but if you divide by the cross-sectional area of the SWNT (1.13×10^{-18} m^2) the equivalent pressure is 0.107 Pascal [N/m^2]. OK, still not much. But make the SWNT 10 times shorter and you will get 1000 times more tension!

3. One-Dimensional Maxwellian Speed Distribution

(*a*) What is the thermal speed distribution $\mathcal{D}(v)$ [in the textbook's notation, $N(v)/N$, as in Eq. (22-14) on p. 503] for an ideal gas confined to a **one**-dimensional "box"? It would be nice if you could find the right leading factors (involving temperature and various constants) to *normalize* the distribution so that

$$\int_0^\infty \mathcal{D}(v)\, dv \;=\; 1\,,$$

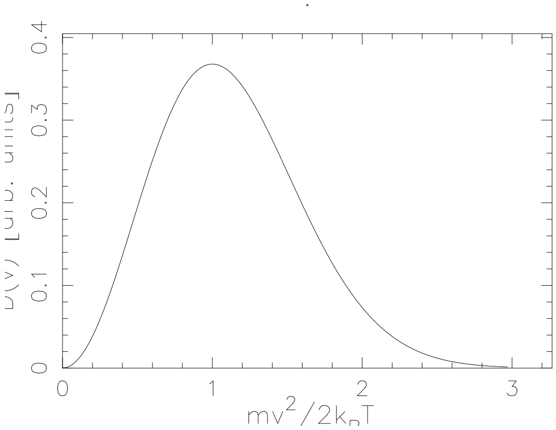
but I am mainly looking for its *dependence on the speed* v.

Hint: Again, use de Broglie's hypothesis and think of *standing waves.*

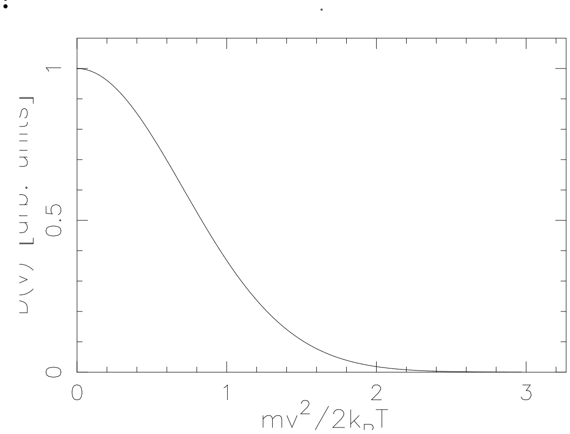
ANSWER: Again the 1D case is much *simpler* than the 3D case because we don't have to worry about composing v^2 out of $v_x^2 + v_y^2 + v_z^2$ (or $\vec{\boldsymbol{k}}$ out of k_x, k_y and k_z). There is just one direction, just one velocity component, one momentum component and one wavelength to worry about making commensurate with the length of the "box". Thus the requirement that an integer n half-wavelengths fit evenly into ℓ gives $\lambda_n = 2\ell/n$ and so $p_n = h/\lambda_n = nh/2\ell = mv_n$ or $v_n = n(h/2m\ell)$. Thus *the possible values of the speed v are evenly spaced every* $(h/2m\ell)$ and the distribution of speeds varies only as the Boltzmann factor $\exp(-\frac{1}{2}mv^2/\tau)$. This gives immediately $\boxed{\mathcal{D}_{1D}(v) = A\, e^{-mv^2/2\tau}}$ where A is a normalization constant that does not depend on v. You get full credit for this result, but here's how to get A: Let $y \equiv mv^2/2\tau$ where τ is treated as a constant. Now, $v^2 = (2\tau/m)y$ or $v = \sqrt{2\tau/m}\, y^{1/2}$ so $dv = \frac{1}{2}\sqrt{2\tau/m}\, y^{-1/2} dy$ and we have $\displaystyle\int_0^\infty \mathcal{D}(v)\, dv \;=\; 1 \;=\; A\sqrt{\frac{\tau}{2m}} \int_0^\infty y^{-1/2}\, e^{-y}\, dy$. You can look up the definite integral; its value is $\sqrt{\pi}$, giving $\boxed{A = \sqrt{2m/\pi\tau}}$.

(*b*) *Sketch* this distribution for a given temperature and compare its shape with that shown in the Figures on p. 503 of the textbook. **ANSWER**: See below.

3D:



1D:



(*c*) What can you say about the *most probable speed* v_p in the two different cases?

ANSWER: As stated in the textbook, the most probable speed for an ideal gas molecule in 3D is $\boxed{v_p^{3D} = \sqrt{2\tau/m}}$ (*i.e.* when $mv^2/2\tau = 1$). For the 1D gas, however, $\mathcal{D}_{1D}(v)$ is missing that extra factor of v^2 that forces its value to zero at $v = 0$, so the resultant speed distribution has its *maximum* at $v = 0$: $\boxed{v_p^{1D} = 0}$.