THE UNIVERSITY OF BRITISH COLUMBIA

Physics 108 Assignment # 4
SOLUTIONS:
GAUSS' LAW

Wed. 26 Jan. 2005 — finish by Wed. 2 Feb.

1. **FIELD WITHIN A UNIFORM CHARGE DISTRIBUTION:** The textbook shows how to use GAUSS' LAW to derive the radial (r) dependence of the electric field $E(r > R)$ *outside* charge distributions of spherical, cylindrical or planar symmetry, where R is the distance the charge distribution extends from the centre of symmetry — the radius of a charged *sphere* or *cylinder*, or half the thickness of an infinite *slab* of charge, respectively. Use similar arguments to show that, for each of these cases (a sphere, cylinder or a slab of uniform charge density), the electric field $E(r < R)$ *inside* the charge distribution is given in terms of the field $E(R)$ *at the boundary* of the charge distribution by $E(r < R) = \left(\frac{r}{R}\right) E(R)$.

ANSWER:

Sphere: charge density $\rho = \dfrac{3Q}{4\pi R^3}$

$$\epsilon_\circ(4\pi r^2 E) = Q_{\rm encl} = \frac{4}{3}\pi r^3 \rho = \left(\frac{r^3}{R^3}\right) Q$$

$$E(r) = \frac{Q}{4\pi\epsilon_\circ}\frac{r}{R^3} = E(R)\left(\frac{r}{R}\right) \quad . \quad \mathcal{QED}$$

Cylinder: $\lambda = \pi R^2 \rho \Longrightarrow \rho = \dfrac{\lambda}{\pi R^2}$

$$\epsilon_\circ(2\pi r L E) = Q_{\rm encl} = \pi r^2 L \rho = L\lambda\left(\frac{r^2}{R^2}\right)$$

$$E(r) = \frac{\lambda}{2\pi\epsilon_\circ}\frac{r}{R^2} = E(R)\left(\frac{r}{R}\right) \quad . \quad \mathcal{QED}$$

Slab: $\sigma = 2R\rho \Longrightarrow \rho = \dfrac{\sigma}{2R}$

$$\epsilon_\circ(2AE) = Q_{\rm encl} = 2Ar\rho = A\sigma\left(\frac{r}{R}\right)$$

$$E(r) = \frac{\sigma}{2\epsilon_\circ}\frac{r}{R} = E(R)\left(\frac{r}{R}\right) \quad . \quad \mathcal{QED}$$

2. **ATOMS AS SPHERES OF CHARGE:** In Rutherford's work on α particle scattering from atomic nuclei, he regarded the atom as having a pointlike positive charge of $+Ze$ at its centre, surrounded by a spherical volume of radius R filled with a uniform charge density that makes up a total charge $-Ze$, making the atom as a whole electrically neutral. In this simple model, calculate the electric field strength E and the electric potential ϕ as functions of radius r and various constants. *Plot* your results for $0 < r \le 2R$. (Choose $\phi \underset{r\to\infty}{\longrightarrow} 0$.)

ANSWER: By Gauss' Law, $\Phi_E \equiv \oint\!\!\oint \vec{E}\cdot d\vec{A} = 4\pi k Q_{\rm enc} = Q_{\rm enc}/\epsilon_\circ$ where $Q_{\rm enc}$ is the total charge inside the closed Gaussian surface: if we use a sphere of radius $r < R$ then $Q_{\rm enc} = +Ze$ (from protons) $-\, Ze(r/R)^3$ (from electrons). By spherical symmetry, $\vec{E}$ points radially outward (normal to the surface) and is the same strength all over the sphere, so Gauss' Law yields $\Phi_E = (E)(4\pi r^2) = Ze\left[1 - (r^3/R^3)\right]/\epsilon_\circ$, giving

$$\boxed{E = \left(\frac{Ze}{4\pi\varepsilon_0}\right)\left(\frac{1}{R^2}\right)\left[\frac{R^2}{r^2} - \frac{r}{R}\right] \text{ for } r < R} \; .$$

At $r = R$ this goes to zero; for $r > R$ there are equal amounts of positive and negative charge enclosed, so Gauss' Law tells us that $\boxed{E(r > R) = 0}$.

Now to calculate the potential ϕ. If we take the potential to be zero for $r \to \infty$, $\boxed{\phi(r > R) = 0}$, then integrating $-E(r')dr'$ from $r' = R$ to $r' = r$ (the same thing as integrating $E(r')dr'$ from $r' = r$ to $r' = R$) gives

$$\phi(r < R) = \left(\frac{Ze}{4\pi\varepsilon_0}\right)\int_r^R \left(\frac{1}{r'^2} - \frac{r'}{R^3}\right) dr' \; .$$

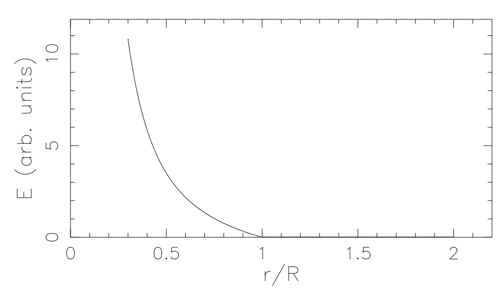
The integral can be split into two parts,

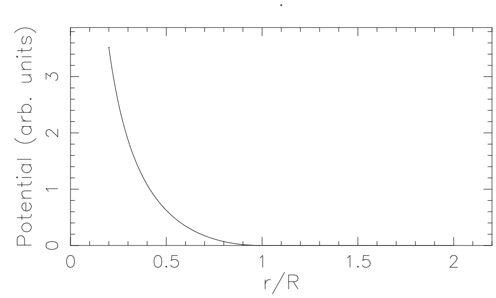
$$\int_r^R \frac{dr'}{r'^2} = \left[-\frac{1}{r'}\right]_r^R = \frac{1}{r} - \frac{1}{R} \qquad \text{and}$$

$$-\int_r^R \frac{r'\,dr'}{R^3} = -\left[\frac{1}{2}\frac{r'^2}{R^3}\right]_r^R = \frac{1}{2}\left\{\frac{r^2}{R^3} - \frac{R^2}{R^3}\right\}$$

Although these are not hard integrals, one can easily fall into confusion by not thinking carefully about the *limits* of integration. Since $R^2/R^3 = 1/R$, we can combine the two results to give

$$\boxed{\phi(r < R) = \left(\frac{Ze}{4\pi\varepsilon_0}\right)\left(\frac{1}{R}\right)\left[\frac{R}{r} + \frac{1}{2}\frac{r^2}{R^2} - \frac{3}{2}\right]} \; .$$





The two graphs have qualitatively similar behaviour — each "blows up" as $r \to 0$ and drops to zero at $r = R$ — but they really are different functions.

3. **TUBEWORLD:**[1] In a future interstellar voyage, you come across a gigantic hollow cylinder a million km long with an outer radius of 1000 km. It is spinning on its axis at an angular frequency ω, but with deft piloting you are able to land your spaceship at what looks like a dock halfway down the cylinder. As you approach, you find that the landing is easy because there is just enough gravitational attraction toward the surface to provide the acceleration you need to be in a circular orbit when you land at the dock. When you leave your craft in a spacesuit, you seem to be weightless. Now you see a door in the cylinder with a button labelled, "Press to open," so you do. A hatch opens and you step through into an airlock; the outer hatch closes, it fills with breathable air at 1 Earth atmosphere and then the inner hatch opens, revealing a hollow interior filled with air and light from a long line source down the axis of the cylinder. As you stand on the inner surface you experience an apparent Earth-normal gravity of "1 g" pulling you toward the surface (away from the axis). Many interesting creatures live here, but before we go meet them I have a few Answers and Questions. First the Answers: Virtually all the mass of the cylinder is in the thin shell through which you just stepped. It is made of a *very* dense material! Although this all requires technology beyond our present grasp,[2] none of it is in conflict with the "known laws of physics" and you have all the tools you need to answer the Questions:

(a) How long does it take for the cylinder to spin once about its axis?

ANSWER: The first observation you must make is that there is no net gravitational force on an object *inside* the cylindrical shell (Gauss' Law). Therefore the "1 g" apparent gravity on the inner surface is entirely the result of centripetal acceleration due to the spin of the cylinder. Thus $g = r\omega^2$ or $\omega = \sqrt{g/r}$. With $g = 9.81$ m/s^2 and $r = 10^6$ m, this gives $\omega = 2\pi/T = 3.132 \times 10^{-3}$ s^{-1} or $\boxed{T = 2006 \text{ s} = 33.43 \text{ minutes.}}$

(b) What is the net mass of the cylinder, and how does it compare with the mass of the Earth? With the mass of the Sun?

ANSWER: If you stand on the *outside* of the cylinder you feel the full gravitational attraction of the cylinder as if it were a "line of mass" along the central axis with a mass density λ_M per unit length. This keeps you in perpetual freefall (*i.e.* in *orbit*) as you spin about the axis at angular frequency ω. It remains only to find $M = \lambda_M L$ (where $L = 10^9$ m) by calculating the λ_M required to produce a gravitational acceleration of $1g$ at a radius of $r = 10^6$ m. Rather than derive the formula for the gravitational acceleration of a line mass from first principles, let's just convert the electrostatic solution to a gravitational one: for *spherical symmetry*, Coulomb's Law ($E = k_E Q/r^2$) differs from Gauss' Law for gravity ($g = GM/r^2$) only by the substitution of g for E and GM for $k_E Q$. Thus we can convert the electrostatic field for *cylindrical symmetry*, $E = 2k_E\lambda/r$, into the desired gravitational version by the same substitution:

$$g = 2G\lambda_M/r$$

giving $\lambda_M = gr/2G$ or $\lambda_M = \frac{9.81\times10^6}{2\times6.67\times10^{-11}} = 7.35 \times 10^{16}$ kg/m. Thus the net mass of this *Tubeworld* is $\boxed{M = \lambda_M L = 7.35 \times 10^{25} \text{ kg}}$ which is more than 10 times heavier than the Earth ($M_E \approx 6 \times 10^{24}$ kg) but only a small fraction of the mass of the Sun ($M_S \approx 2 \times 10^{30}$ kg). The construction of such a habitat is therefore an impressive engineering project but not an implausible one for that "sufficiently advanced technology".

[1] The *Ringworld* novels of Larry Niven inspired this adaptation including gravity and a more enclosed geometry.

[2] [Arthur C.]Clarke's Law: "Any sufficiently advanced technology is indistinguishable from magic."