THE UNIVERSITY OF BRITISH COLUMBIA

Physics 108 Assignment # 5 SOLUTIONS: POTENTIAL & CAPACITANCE

Wed. 2 Feb. 2005 — finish by Wed. 9 Feb.

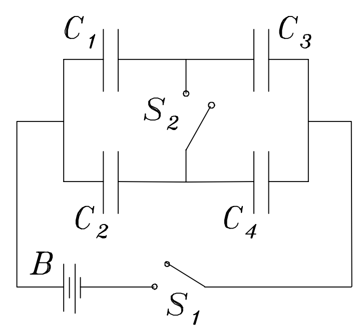
1. CLASSICAL RADIUS OF THE ELECTRON: You are probably familiar with Einstein's famous equation $E = mc^2$. If m is the mass of an electron and E is the electrostatic potential energy required to "assemble" the electron from bits of charge infinitely distant from each other into a uniform spherical shell of radius r_0 and net charge e, find the numerical value of r_0 in meters.[1]

ANSWER: Start with no charge, then bring successive bits dQ in to add uniformly to the charge Q of the shell. A given bit of charge acquires an electrostatic potential energy $dE = k_E Q dQ/r_0$ in the process. Thus the total energy E required to assemble the shell is $E = k_E/r_0 \int_0^e Q dQ = \frac{1}{2} k_E e^2/r_0$. If we set this equal to mc^2 we get $r_0 = \frac{1}{2} k_E \frac{e^2}{mc^2} = \frac{(8.998 \times 10^9)(1.6022 \times 10^{-19})^2}{2(9.11 \times 10^{-31})(2.998 \times 10^8)^2}$ or $\boxed{r_0 = 1.409 \times 10^{-15} \text{ m}}$. The Compton wavelength of the electron is twice as big, 2.818 fm, where "fm" stands for "femtometers" or "fermis" (named after Enrico Fermi); both are the same as 10^{-15} m.

2. CAPACITOR WITH INSERT: Suppose we have a capacitor made of two large flat parallel plates of the same area A (and the same shape), separated by an air gap of width d. Its capacitance is C. Now we slip another planar conductor of width $d/2$ (and the same area and shape) between the plates so that it is centred halfway in between. What is the capacitance C' of the new system of three conductors, in terms of the capacitance C of the original pair and the other parameters given? (Neglect "edge effects" and any dielectric effect of air.)

ANSWER: The original capacitance was $C = \epsilon_\circ A/d$. The capacitor with the insert is equivalent to two identical capacitors in series, each of which has a gap of $d/4$ between its plates, so that $C_1 = C_2 = 4C$. The equivalent capacitance of two capacitors in series is given by $1/C' = 1/C_1 + 1/C_2 = 2/4C = 1/2C$. Thus $\boxed{C' = 2C}$.

3. ARRAY of CAPACITORS: The battery B supplies 12 V. The capacitances are $C_1 = 1\ \mu$F, $C_2 = 2\ \mu$F, $C_3 = 4\ \mu$F and $C_4 = 3\ \mu$F.



(a) Find the charge on each capacitor when switch S_1 is closed but switch S_2 is still open. ANSWER: Let Q_i denote the charge on the i^{th} capacitor C_i. From charge conservation we have $Q_1 = Q_3$ and $Q_2 = Q_4$. Both pairs of capacitors in series (1 and 3; 2 and 4) must make up the full voltage: $V_B = Q_1/C_1 + Q_3/C_3 = Q_2/C_2 + Q_4/C_4$. Therefore $V_B = Q_1[1/C_1 + 1/C_3] = Q_2[1/C_2 + 1/C_4]$ yielding $Q_1 = Q_3 = 12/(10^6/1 + 10^6/4)$ and $Q_2 = Q_4 = 12/(10^6/2 + 10^6/3)$ or $\boxed{Q_1 = Q_3 = 9.6 \times 10^{-6} \text{ C}}$ and $\boxed{Q_2 = Q_4 = 14.4 \times 10^{-6} \text{ C}}$.

(b) What is the charge on each capacitor if S_2 is also closed? **ANSWER**: Now C_1 and C_2 are effectively just one big capacitor $C_{12} = C_1 + C_2 = 3\ \mu$F and similarly for $C_{34} = C_3 + C_4 = 7\ \mu$F. Charge conservation now requires $Q_{12} \equiv Q_1 + Q_2 = Q_{34} \equiv Q_3 + Q_4$ and the two effective capacitors in series must make up the full voltage: $V_B = Q_{12}/C_{12} + Q_{34}/C_{34}$. Thus $V_B = Q_{12}[1/C_{12} + 1/C_{34}]$, giving $Q_{12} = Q_{34} = 12/(10^6/3 + 10^6/7)$ $= 25.2 \times 10^{-6}$ C. Meanwhile the voltage across C_1 must be the same as that across C_2: $Q_1/C_1 = Q_2/C_2$ $\Longrightarrow Q_2 = Q_1(C_2/C_1) = \frac{2}{1} Q_1 \Longrightarrow Q_{12} = \left(1 + \frac{2}{1}\right) Q_1$ or $\boxed{Q_1 = \frac{1}{3} Q_{12} = 8.4 \times 10^{-6} \text{ C}}$ and $\boxed{Q_2 = Q_{12} - Q_1 = 16.8 \times 10^{-6} \text{ C}}$. Similarly, $Q_3/C_3 = Q_4/C_4 \Longrightarrow Q_4 = Q_3(C_4/C_3) = \frac{3}{4} Q_3$ $\Longrightarrow Q_{34} = \left(1 + \frac{3}{4}\right) Q_3$ or $\boxed{Q_3 = \frac{4}{7} Q_{34} = 14.4 \times 10^{-6} \text{ C}}$ and $\boxed{Q_4 = Q_{34} - Q_3 = 10.8 \times 10^{-6} \text{ C}}$.

[1] The value you calculate will not agree with the value you look up; this is because the r_0 listed in textbooks is actually the Compton radius of the electron and has a completely different meaning. Nevertheless, numerous texts glibly describe r_0 as defined in this problem. The amazing thing is that the two values are so close!

4. THUNDERCLOUD CAPACITOR: A large thundercloud hovers over the city of Vancouver at a height of 1.0 km. Between the cloud and the ground (both of which we may treat as parallel conducting plates, neglecting edge effects) the electric field is about 300 V/m. The cloud has a horizontal area of 100 km^2.

(*a*) Estimate the number of Coulombs [C] of positive charge in the cloud, assuming that the ground has the same surface density of negative charge.

ANSWER: The electric field between two flat plates with surface charge densities $\pm\sigma$ is given by $E = \sigma/\epsilon_\circ$. Thus $\sigma = \epsilon_\circ E = 8.85 \times 10^{-12} \times 300 = 2.656 \times 10^{-9}$ C/m^2. Over an area of $A = 100 \times 10^6 = 10^8$ m^2, this gives a total charge of $\boxed{Q = \sigma A = 0.2656 \text{ C}}$.

(*b*) Estimate the number of joules [J] of energy contained in the air between the cloud and the ground.

ANSWER: The energy density stored in an electric field is given by $U/V = \frac{1}{2}\epsilon_\circ E^2$ $= 0.5 \times 8.85 \times 10^{-12} \times (300)^2 = 3.984 \times 10^{-7}$ J/m^3. The volume between the cloud and the ground is $V = 1000 \times 10^8 = 10^{11}$ m^3, so $\boxed{U = 3.984 \times 10^4 \text{ J}}$.