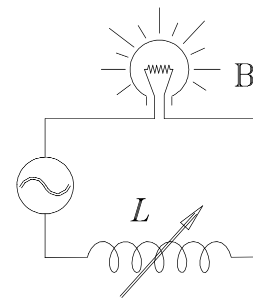
The University of British Columbia

Physics 108 Assignment # 10 SOLUTIONS:

AC CIRCUITS & EM WAVES

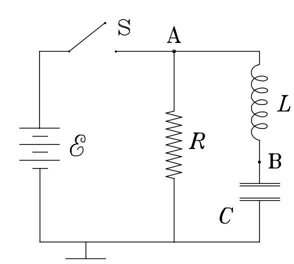
Wed. 16 Mar. 2005 — finish by Wed. 23 Mar.



1. DIMMER SWITCH: A typical "light dimmer" switch consists of a variable inductor L connected in series with the light bulb B as shown in the diagram on the right. The power supply produces 120 V (rms) at 60.0 Hz and the light bulb is marked "120 V, 100 W." Assume that the light bulb is a simple resistor whose resistance does not depend on its temperature.

(a) What maximum inductance L is required if the power in the light bulb is to be varied by a factor of five? **ANSWER**: Since we are looking for a given power *ratio*, the answer will be independent of the rms driving voltage (see below). The frequency is $\omega = 2\pi(60\text{ Hz}) = 377\text{ s}^{-1}$. The light bulb draws $P = \mathcal{E}^2/R = 100$ W when the voltage drop across it is $\mathcal{E} = 120$ V, so $R = \mathcal{E}^2/P = (120)^2/100 = 144\ \Omega$. Now, for an LR circuit the impedance is just $Z = \sqrt{R^2 + X_L^2} = \sqrt{R^2 + \omega^2 L^2}$ (*i.e.* we just omit the capacitive reactance term)[1] so that $I_m = \mathcal{E}_m/Z$ and the average power dissipated in the resistance (light bulb) is $\overline{P} = \frac{1}{2} I_m^2 R = \frac{R}{2} \cdot \frac{\mathcal{E}_m^2}{Z^2} = \frac{R}{2} \cdot \frac{\mathcal{E}_m^2}{R^2 + \omega^2 L^2}$. When $L \to 0$ we get a maximum value of $\overline{P}(\max) = \frac{\mathcal{E}_m^2}{2R}$ and when $L \to L_{\max}$ we get a minimum value of $\overline{P}(\min) = \frac{R}{2} \cdot \frac{\mathcal{E}_m^2}{R^2 + \omega^2 L_{\max}^2}$. We want the ratio of these to be five: $\frac{\overline{P}(\max)}{\overline{P}(\min)} = \frac{R^2 + \omega^2 L_{\max}^2}{R^2} = 5 \implies R^2 + \omega^2 L_{\max}^2 = 5R^2 \implies \omega^2 L_{\max}^2 = 4R^2$ or $\omega L_{\max} = 2R$ giving $L_{\max} = \frac{2R}{\omega} = \frac{2 \times 144}{377}$ or $\boxed{L_{\max} = 0.764\text{ H}}$.

(b) Could one use a variable resistor in place of the variable inductor? If so, what maximum resistance R would be required? Why isn't this done? (Variable resistors [rheostats] are generally cheaper than variable inductors.) **ANSWER**: The circuit would be cheaper to build using a rheostat in place of the variable inductance and it would work fine: the two resistances in series would just add up to one net resistance $R_{\text{tot}} = R_{\text{light}} + R_{\text{var}}$ so that $I_{rms} = \mathcal{E}_{rms}/R_{\text{tot}}$ and $\overline{P}_{\text{light}} = I_{rms} R_{\text{light}}^2 = \mathcal{E}_{rms} R_{\text{light}}^2/(R_{\text{light}} + R_{\text{var}})$, which can be varied from a minimum of $\mathcal{E}_{rms} R_{\text{light}}^2/(R_{\text{light}} + R_{\max})$ to a maximum of $\mathcal{E}_{rms} R_{\text{light}}$, the ratio of which $[R_{\text{light}}/(R_{\text{light}} + R_{\max})]$ can easily be made equal to 5. The problem with this simpler design is that when the light is *dimmed* ($\overline{P}_{\text{light}}$ minimized), considerable power $[I_{rms} R_{\text{var}}^2]$ is being uselessly dissipated in the rheostat! A pure inductance, by contrast, does not draw any power; it merely shifts the phase of the voltage and current.



2. *LCR* CIRCUIT TIME-DEPENDENCE: In the circuit shown, the battery has negligible internal resistance. The switch **S** is closed for a long time, then opened. Describe *qualitatively* what happens in the circuit after the switch is closed and then after it is opened again, for two cases:

(a) $R/2L > 1/\sqrt{LC}$ and

(b) $R/2L < 1/\sqrt{LC}$.

ANSWER: This problem has a tricky part, namely the behaviour <u>when the switch is first closed</u>. In this situation $\sum \Delta\mathcal{E} = 0$ around the leftmost loop requires that a current $I_1 = \mathcal{E}/R$ immediately start flowing through R and continue that way forever. So, for all practical purposes, R simply serves to drain the battery and has no effect at all on the voltages in the outer loop, which consists of a battery driving an undamped LC circuit — which will therefore $\boxed{\text{oscillate indefinitely}}$ at its resonant frequency $\omega_0 = 1/\sqrt{LC}$ without any damping at all, regardless of the size of R!

[1]This can be shown rigourously without too much difficulty, but it makes sense that a term due to a nonexistent component should disappear from the formula.

When the switch is opened again, we do get different behaviour depending on the ratio of R to L. Specifically, if we define $\lambda = R/2L$ and $\omega_0 = 1/\sqrt{LC}$, for $\boxed{\lambda > \omega_0}$ we have an "overdamped" oscillator in which the current will decay away exponentially $\boxed{\text{with no oscillations}}$, whereas for $\boxed{\lambda < \omega_0}$ we will see $\boxed{\text{oscillations}}$ at the frequency $\boxed{\omega = \sqrt{\omega_0^2 - \lambda^2}}$ whose $\boxed{\text{amplitude decays exponentially}}$ with a time constant $\tau = 1/\lambda$.

3. ELECTROMAGNETIC WAVE: The electric field associated with a plane electromagnetic wave is given by

$$E_x = 0, \quad E_y = 0 \quad \text{and} \quad E_z = E_0 \sin[k(x - ct)],$$

where $E_0 = 2.34 \times 10^{-4}$ V/m and $k = 9.72 \times 10^6$ m^{-1}. The wave is propagating in the $+x$ direction.

(*a*) Write expressions for all three components of the magnetic field associated with the wave.
ANSWER: The wave propagates in the direction given by $\vec{E} \times \vec{B}$ (see textbook's section on the POYNTING VECTOR) and so, since $\vec{E}$ points in the $\hat{k}$ direction and the wave propagates in the $\hat{\imath}$ direction, we must have $\hat{\imath} = \hat{k} \times \hat{B}$ where $\hat{B}$ is the unit vector in the $\vec{B}$ direction. The only unit vector satisfying this criterion is $\hat{B} = -\hat{\jmath}$ and so $\vec{B}$ is in the negative y direction when $\vec{E}$ is in the positive z direction. Both must oscillate sinusoidally with the same wavelength and frequency, so they share the same propagation speed c, the same wavenumber k and the same angular frequency $\omega = ck$. This allows us to write down the answer:
$\boxed{B_x = B_z = 0 \quad \text{and} \quad B_y = -B_0 \sin[k(x - ct)]}$.
Having established this, we need only find the relationship between the *amplitudes* of the E and B fields. Their *magnitudes* must satisfy $\left|\frac{\partial E}{\partial x}\right| = \left|\frac{\partial B}{\partial t}\right|$, giving $kE_0 = \omega B_0$ or $B_0 = \frac{k}{\omega} E_0$ or
$\boxed{B_0 = E_0/c = \frac{2.34\times10^{-4}}{2.997\times10^{8}} = 0.7805 \times 10^{-12} \text{ T}}$. (This also follows from $E = cB$ [always valid].)

(*b*) Find the wavelength of the wave. **ANSWER**: All you need for this part is the universal relationship

$$k \equiv \frac{2\pi}{\lambda} \quad \Longrightarrow \quad \lambda = \frac{2\pi}{k} = \frac{2\pi}{9.72 \times 10^6}$$

or $\boxed{\lambda = 0.6464 \times 10^{-6} \text{ m} = 0.6464\ \mu\text{m}}$.

NOTE: This was a rather trivial problem, designed partly to remind you of the properties of *waves*, our subject for most of the rest of the course. I have been rather longwinded in the solutions; you may be much more economical with words, but be sure you always make your basic reasoning clear!