THE UNIVERSITY OF BRITISH COLUMBIA

Physics 108 Assignment # 12 SOLUTIONS:

DIFFRACTION

Wed. 30 Mar. — finish by Wed. 6 Apr. 2005

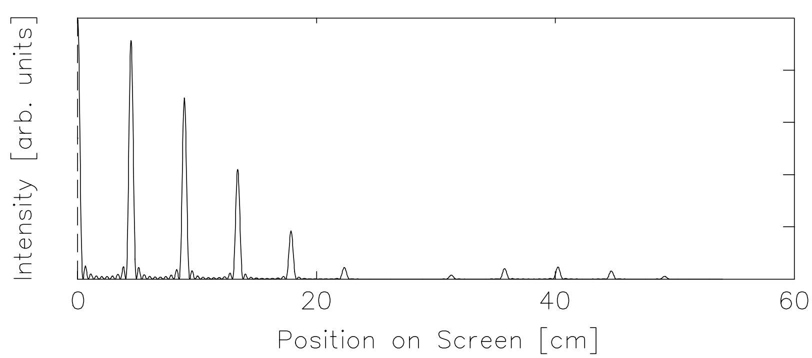
1. SOUND DIFFRACTION: You are standing 30 m away from a large, soundproof building, directly in front of a 1 m wide doorway, when it opens and you hear someone blowing a loud whistle at a frequency of 1320 Hz from somewhere directly behind the door, deep inside the building. How far sideways (parallel to the wall with the door in it) must you move before the sound first becomes imperceptible? [Assume that the speed of sound is 343 m/s.]
ANSWER: The wavelength λ of the sound waves is given by $\lambda = c/\nu = 343/1320 = 0.260$ m, where $c = 343$ m/s. The door ("slit") is narrow compared with its distance from the source deep inside the building, so we may treat it as uniformly illuminated by plane waves. The sound will first become imperceptible at the position of the first minimum of the diffraction pattern. Thus we need to find $x_1 = D\tan\theta_1$, where $D = 30$ m. This is where $I = I_0\left(\frac{\sin\alpha}{\alpha}\right)^2 \to 0$, *i.e.* where $\alpha \equiv \pi\frac{a}{\lambda}\sin\theta = \pi$ or where $a\sin\theta_1 = \lambda$, giving $\sin\theta_1 = \lambda/a = 0.260/1.0 = 0.260$ or $\theta_1 = 0.263$ radian (15.06°), not a particularly small angle. Thus $\boxed{x_1 = D\tan\theta_1 = 8.7 \text{ m}}$ is the position in question.

2. 10-SLIT MINI-GRATING: Consider a series of 10 equal slits, each 0.004 cm wide, arrayed in a straight line with equal spacing so that 0.22 cm separates the centres of the first and last slits. This configuration is illuminated from behind by a distant source of light of wavelength 0.546 microns [μm].

(*a*) How many *major* intensity peaks will you see in the *entire* pattern displayed on a wide screen 20 m away before the pattern first fades away laterally because of the diffraction from each aperture? **ANSWER**: The first minimum of the DIFFRACTION envelope occurs at an angle θ_1^D given by $a\sin\theta_1^D = \lambda$; meanwhile, the principal maxima of the N-slit INTERFERENCE pattern occur at angles θ_m^I given by $d\sin\theta_m^I = m\lambda$. The question then becomes, "What is the largest m for which $\sin\theta_m^I \le \sin\theta_1^D$?" If we treat $\tilde{m}$ as a continuous variable, we can solve for $\tilde{m}$ by setting $\theta_m^I = \theta_1^D$: $\tilde{m}\lambda/d = \lambda/a$ or $\tilde{m} = d/a$. In this case, $a = 4\times10^{-5}$ m and $d = (0.22/9)\times10^{-2} = 2.444\times10^{-4}$ m, so that $\tilde{m} = 2.444\times10^{-4}/4\times10^{-5} = 6.111$ and thus the largest *integer* m is $m_{\max} = 6$, giving $\boxed{2m_{\max}+1 = 13}$ peaks in the region specified. Note that neither the distance to the screen nor the wavelength of the light (as long as it is small compared to a) is relevant to the answer.

(*b*) How far apart and how wide[1] (in mm) are the principal[2] maxima near the centre of the screen?
ANSWER: The angles θ_m^I at which principal maxima occur are given by $d\sin\theta_m^I = m\lambda$. Thus the angular separation $\Delta\theta$ between the central maximum and the first principal maximum of the interference pattern is given by $\sin\Delta\theta = \lambda/d = 5.46\times10^{-7}/2.444\times10^{-4} = 2.23\times10^{-3}$ or $\Delta\theta \simeq 2.23\times10^{-3}$ radians. (The small-angle approximation is excellent here.) The width of each principal maximum is $1/N$ of the distance between them: $\delta\theta = \Delta\theta/N$. Here $N = 10$, giving $\delta\theta = 2.23\times10^{-4}$ radians. Now, to translate this into positions on a screen $D = 20$ m away we must multiply D by $\tan\theta \simeq \theta$, giving a distance $\boxed{\Delta x = 0.04467 \text{ m} = 4.467 \text{ cm}}$ between principal maxima of width $\boxed{\delta x = 0.4467 \text{ cm}}$.

(*c*) *Sketch* the intensity pattern from the central maximum out to the second diffraction minimum on one side.
ANSWER: You now have everything you need to make the sketch:



[1] [The *width* is the distance from the nearest zero on one side to the nearest zero on the other side.]
[2] ["principal" ≡ "major"]

3. DIFFRACTION-LIMITED VISION: In fairly bright light the pupil of your eye will contract to a diameter of about 4 mm. Under these conditions, assuming that you have "perfect" vision,

(a) how far from your eye can you hold a book and still be able to resolve two black dots separated by 100 μm, illuminated by yellow light with $\lambda = 575$ nm? **ANSWER**: The RESOLUTION FUNCTION $\mathcal{R}(r)$ of a DETECTOR receiving a signal from a point source at distance D is the same thing as the INTENSITY distribution $\mathcal{I}(r)$ from a SOURCE of the same size and shape at that distance. The dots are resolvable if the first minimum of $\mathcal{R}(r)$ centred on one dot coincides with the location of the second dot a transverse distance x away. That is (by the RAYLEIGH CRITERION), if $D \tan\theta_1 = x$ where $\sin\theta_1 = 1.22\lambda/a$, $\lambda = 0.575$ μm and $a = 4000$ μm is the diameter of the circular aperture (your pupil). This gives $\theta_1 \simeq \sin\theta_1 = 1.754 \times 10^{-4}$ radian, corresponding to $x = 100$ μm at a distance D given by $x = D\tan\theta_1 \simeq D\theta_1$ or $D = x/\theta_1$, in this case $\boxed{D = 0.57 \text{ m}}$.

(b) Describe what you would see at this distance when observing a *single white dot* on a *black page* illuminated with *white* light (all wavelengths). **ANSWER**: The analysis above predicts that *yellow* light from such a dot at that distance will produce a diffraction pattern on your retina that makes the dot look smeared out over a radius $x \approx 100$ μm on the page. This radius will be proportionally larger for longer wavelengths (red light) and smaller for shorter wavelengths (blue light). All wavelengths will have a central maximum at the same place, so the centre of the dot will appear white; but the blue wavelengths drop off at shorter radii, leaving a reddish halo growing redder and dimmer as the blue components reach their first minimum at about $x_{\text{blue}} \approx 100 \times 400/575 \approx 70$ μm, after which the red component continues to decrease toward $x_{\text{red}} \approx 100 \times 700/575 \approx 108$ μm, at which point the blue has partially recovered; and so on. Of course, the intensity drops off rapidly away from the central maximum, so all you can really hope to see is a white centre with a reddish halo about 140 μm in diameter surrounded by a fainter bluish halo at nearly twice that diameter.