THE UNIVERSITY OF BRITISH COLUMBIA

Physics 401 Assignment # 6: Electromagnetic Waves SOLUTIONS:

Wed. 8 Feb. 2006 — finish by Wed. 22 Feb.

1. **CMBR:** Most of the electromagnetic energy in the universe is in the cosmic microwave background radiation (CMBR), sometimes referred to as the 3° Kelvin background. Penzias and Wilson discovered the CMBR in 1965 using a radio telescope, and subsequently received the Nobel Prize for this discovery. This background radiation has wavelength $\lambda \sim 1.1$ mm. The energy density of the CMBR is about 4.0×10^{-14} J/m^3. What is the *rms* electric field strength of the CMBR?
ANSWER: If $\langle u_{EM} \rangle = \epsilon_0 \langle E^2 \rangle = 4.0 \times 10^{-14}$ J/m^3, then $E_{rms} \equiv \sqrt{\langle E^2 \rangle} = \sqrt{\langle u_{EM} \rangle / \epsilon_0} = \sqrt{\frac{4.0\times10^{-14}}{0.88541878\times10^{-11}}} = \boxed{0.0672 \text{ V/m}}$. (The wavelength, while interesting, is irrelevant to the question.)

2. **STANDING WAVES:** Consider standing electromagnetic waves:

$$\vec{E} = E_0 (\sin kz \ \sin \omega t)\, \hat{x} \qquad \text{with}$$
$$\vec{B} = B_0 (\cos kz \ \cos \omega t)\, \hat{y} \; .$$

(*a*) Show that these satisfy the wave equation (9.2). **ANSWER**: When we're taking the spatial derivatives, the t-dependent factor is just part of the amplitude, and vice versa. Thus $\nabla^2 \sin kz = -k^2 \sin kz$ and $\nabla^2 \cos kz = -k^2 \cos kz$; $\partial/\partial t \sin \omega t = -\omega^2 \sin \omega t$ and $\partial/\partial t \cos \omega t = -\omega^2 \cos \omega t$; so $\nabla^2 \vec{E} - (1/c^2)\partial \vec{E}/\partial t = (-k^2 + \omega^2/c^2)\vec{E}$ and similarly for $\vec{B}$. But $(-k^2 + \omega^2/c^2) = -k^2[1 - (\omega^2/k^2)/c^2] = 0$, since $c = \omega/k$. Thus $\boxed{\Box^2 \vec{E} = 0}$ and similarly for $\vec{B}$. ✓ $\mathcal{QED}$

(*b*) Show that we must also have $c = \omega/k$ and $E_0 = cB_0$. **ANSWER**: Since $c = \omega/k$ is a universal property of all solutions of The Wave Equation (TWE), that's a given. Applying FARADAY'S LAW, $\vec{\nabla} \times \vec{E} = -\partial \vec{B}/\partial t$, gives $E_0 \sin \omega t \ \vec{\nabla} \times (\sin kz \ \hat{x}) = -B_0 \cos kz \ \partial(\cos \omega t)/\partial t \ \hat{y}$ or $k \ \hat{y} \ E_0 \cos kz \ \sin \omega t = \omega \ \hat{y} \ B_0 \cos kz \ \sin \omega t$. Dividing out the common factor $\hat{y} \ \cos kz \ \sin \omega t$ gives $kE_0 = \omega B_0$ or (since $c = \omega/k$) $\boxed{E_0 = cB_0}$. ✓ $\mathcal{QED}$

(*c*) Show that the time-averaged power flow across *any* area will be zero.
ANSWER: $\vec{S} = \vec{E} \times \vec{H} = (\hat{x} \times \hat{y}) (E_0 B_0/\mu) (\sin kz \ \sin \omega t) (\cos kz \ \cos \omega t) = \hat{z} \ (E_0 B_0/\mu) (\sin kz \ \cos kz) (\sin \omega t \ \cos \omega t)$. Looking only at the t-dependence to get the time average, we note that $\sin \omega t \ \cos \omega t = \frac{1}{2} \sin(2\omega t)$ which averages to zero. ✓ $\mathcal{QED}$

(*d*) Show that the Poynting vector will also be zero, *i.e.* there is no net energy flow.
ANSWER: I must apologize for a defective question. [The hazards of using someone else's problem!] As explained above, $\vec{S} = (E_0 B_0/4\mu) \ \sin 2kz \cdot \sin 2\omega t \ \hat{z}$. This is only zero where $\sin 2kz = 0$, *i.e.* at $z = 0$ and $2kz = n\pi$ (where n is any integer). That is, for $z = n\lambda/4$. At any other position, $\vec{S}$ oscillates in the $\pm\hat{z}$ direction, averaging to zero.

3. (p. 386, Problem **9.14**) — **REFLECTED & TRANSMITTED POLARIZATION:** In Eqs. (9.76) and (9.77) it was tacitly assumed that the reflected and transmitted waves have the same *polarization* as the incident wave, namely along the $\hat{x}$ direction. Prove that this *must* be so. [*Hint:* Let the polarization vectors of the reflected and transmitted waves be

$$\hat{n}_T = \cos\theta_T \hat{x} + \sin\theta_T \hat{y} \qquad \text{and}$$
$$\hat{n}_R = \cos\theta_R \hat{x} + \sin\theta_R \hat{y}$$

and prove from the boundary conditions that $\theta_T = \theta_R = 0$.] **ANSWER**: We must have $\vec{E}_\parallel$ continuous across the boundary. Since the normal direction is $\hat{k} = \hat{z}$, $\vec{E}_\parallel$ is constituted of x and y components. Thus $\vec{E}_I + \vec{E}_R = \vec{E}_T$ or $E_I + E_R \cos\theta_R = E_T \cos\theta_T$ [1] and $E_R \sin\theta_R = E_T \sin\theta_T$ [2]. Similarly, $\vec{H}_\parallel$ must be continuous across the boundary, and, as always, $v\vec{B} = \hat{k} \times \vec{E}$, giving $\frac{E_I - E_R \cos\theta_R}{\mu_1 v_1} = \frac{E_T \cos\theta_T}{\mu_2 v_2}$ [3] and $\frac{E_R \sin\theta_R}{\mu_1 v_1} = -\frac{E_T \sin\theta_T}{\mu_2 v_2}$ [4]. If $\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2}$, Eq. [4] reads $E_R \sin\theta_R = -\beta E_T \sin\theta_T$, which we can combine with Eq. [2] to conclude that $E_T \sin\theta_T = -\beta E_T \sin\theta_T$, which can be true only if $E_T = 0$ (trivial case) or $\boxed{\theta_T = 0}$ (mod 2π). Equation [2] then also requires $\boxed{\theta_R = 0}$ (mod 2π). ✓ $\mathcal{QED}$

4. (p. 392, Problem **9.15**) — **COMPLEX ALGEBRA EXERCISE:** Suppose that we

have six nonzero constants A, B, C, a, b, c such than $Ae^{iax} + Be^{ibx} = Ce^{icx}$ for all x. Prove that $a = b = c$ and $A + B = C$. **ANSWER**: The first part is easy: if it were *not* true that $a = b = c$ then even if the equation were satisfied at some position in x, it would *not* be satisfied at some nearby x. So $a = b = c$. ✓ The second part is even easier: at $x = 0$, $A + B = C$. Done. ✓

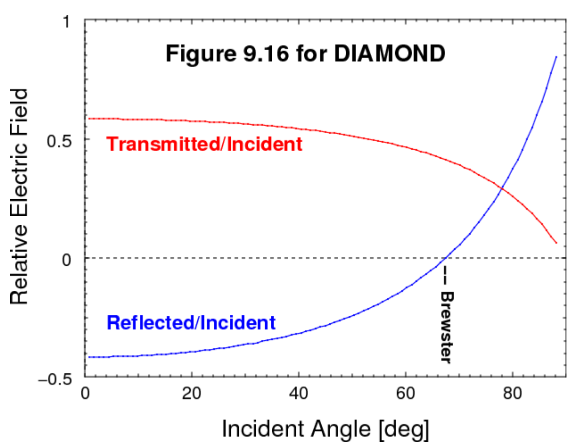
5. (p. 392, Problem **9.17**) — **DIAMOND:** The index of refraction of diamond is 2.42. Construct the graph analogous to Figure 9.16 for the air/diamond interface. (Assume $\mu_1 = \mu_2 = \mu_0$.) **ANSWER**: FRESNEL'S EQUATIONS read

$$\frac{\tilde{E}_0^R}{\tilde{E}_0^I} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right), \quad \frac{\tilde{E}_0^T}{\tilde{E}_0^I} = \left(\frac{2}{\alpha + \beta}\right)$$

where

$$\alpha \equiv \frac{\cos\theta_T}{\cos\theta_I} = \frac{\sqrt{1 - \sin^2\theta_T}}{\cos\theta_I} = \frac{\sqrt{1 - \left[\frac{n_1}{n_2}\sin\theta_I\right]^2}}{\cos\theta_I}$$

and $\beta \equiv \dfrac{\mu_1 v_1}{\mu_2 v_2} = \dfrac{\mu_1 n_2}{\mu_2 n_1}$. In this case $\beta = 2.42$ (we assume the light is *entering* the diamond rather than emerging) and $\alpha = \dfrac{\sqrt{1 - (\sin\theta_I/2.42)^2}}{\cos\theta_I}$.
You can use your favourite spreadsheet or other plotting software to produce the graph below. (I used `http://musr.org/muview/`, a free *Java* spreadsheet applet we built at TRIUMF.)



In particular, calculate

(a) the amplitudes at normal incidence;
ANSWER: For $\theta_I = 0$, $\alpha = 1$, giving $\tilde{E}_0^R = \dfrac{1 - 2.42}{1 + 2.42}\tilde{E}_0^I$ or $\boxed{\tilde{E}_0^R = -0.4152\tilde{E}_0^I}$
and $\tilde{E}_0^T = \dfrac{2}{1 + 2.42}\tilde{E}_0^I$ or
$\boxed{\tilde{E}_0^T = 0.5848\tilde{E}_0^I}$.

(b) Brewster's angle;
ANSWER: $\sin^2\theta_B = \dfrac{1 - \beta^2}{(n_1/n_2)^2 - \beta^2}$
$= \dfrac{1 - 5.8564}{(1/5.8564) - 5.8564} = 0.85415$ or
$\sin\theta_B = 0.9242 \Rightarrow \boxed{\theta_B = 67.55°}$.

(c) and the "crossover" angle at which the reflected and transmitted amplitudes are equal. **ANSWER**: Rather than try to read this off the graph, let's calculate it exactly: The condition is $\alpha - \beta = 2$ or
$\alpha = \dfrac{\sqrt{1 - (\sin\theta_I/2.42)^2}}{\cos\theta_I} = 4.42$ or
$1 - (\sin\theta_I/2.42)^2 = 19.5364\cos^2\theta_I$ or
$5.8564 - 1 + \cos^2\theta_I = 114.413\cos^2\theta_I$ or
$4.8564 = 113.413\cos^2\theta_I$ or
$\cos^2\theta_I = 4.8564/113.413 = 0.04282$ or
$\cos\theta_I = 0.20693 \Rightarrow \boxed{\theta_C = 78.06°}$.

6. **PLANE WAVE STRESS TENSOR:** Find all the elements of the Maxwell stress tensor of a monochromatic plane wave traveling in the z-direction, polarized in the x-direction:

$$\vec{\boldsymbol{E}}(z,t) = E_0\cos(kz - \omega t + \delta)\hat{\boldsymbol{x}}$$

$$\vec{\boldsymbol{B}}(z,t) = \frac{E_0}{c}\cos(kz - \omega t + \delta)\hat{\boldsymbol{y}}$$

ANSWER: Recall Eq. (8.19) on p. 352:

$$T_{ij} = \epsilon_0\left(E_iE_j - \delta_(ij)E^2/2\right) + \left(B_iB_j - \delta_(ij)B^2/2\right)/\mu_0 .$$

Here $E_i = \delta_{i1}E$ where $E \equiv E_0\cos(kz - \omega t + \delta)$ and $B_i = \delta_{i2}B$ where $B \equiv \frac{E_0}{c}\cos(kz - \omega t + \delta)$ $= E/c$, so all off-diagonal elements are zero. We have $T_{11} = \epsilon_0\left(E^2 - E^2/2\right) - B^2/2\mu_0 =$ $\epsilon_0\left(E^2/2 - E^2/2\epsilon_0\mu_0c^2\right) = \epsilon_0\left(E^2/2 - E^2/2\right)$ or $T_{11} = 0$, $T_{22} = -\epsilon_0E^2/2 + \left(B^2 - B^2/2\right)\mu_0 =$ $\epsilon_0\left(-E^2/2 + E^2/2\epsilon_0\mu_0c^2\right) = \epsilon_0\left(-E^2/2 + E^2/2\right)$ or $T_{22} = 0$ and $T_{33} = -\epsilon_0E^2/2 - B^2/2\mu_0$ or (only nonzero element!) $\boxed{T_{33} = -\epsilon_0E^2 = -u_{EM}}$.

In what direction does this EM wave transport momentum? Does this agree with the form of the Maxwell stress tensor you just deduced?
ANSWER: If T_{ij} represents the force per unit area acting in the $\hat{\boldsymbol{x}}_i$ direction on a surface whose normal is in the $\hat{\boldsymbol{x}}_j$ direction, then the diagonal elements are *pressures* and T_{33} is the radiation pressure on a surface normal to $\hat{\boldsymbol{z}}$. In the same way $-T_{33}$ represents the the momentum current density transported by the fields, and is (as expected) in the same direction as $\hat{\boldsymbol{k}}$ and is, in fact, equal to $\vec{\boldsymbol{S}}/c$.

7. (p. 412, Problem **9.33**) — **SPHERICAL WAVES:** Suppose that

$$\vec{\boldsymbol{E}}(r,\theta,\phi,t) = \frac{A\sin\theta}{r}\left[\cos\left(kr-\omega t\right) - \left(\frac{1}{kr}\right)\sin\left(kr-\omega t\right)\right]\hat{\boldsymbol{\phi}}$$

with $c = \omega/k$, as usual. [This is, incidentally, the simplest possible **spherical wave**. For notational convenience, let $(kr - \omega t) \equiv u$ in your calculations.]

(a) Show that $\vec{\boldsymbol{E}}$ obeys all four of Maxwell's equations, in vacuum, and find the associated magnetic field.
ANSWER: Since $\vec{\boldsymbol{E}} = E\hat{\boldsymbol{\phi}}$ and E does not depend on ϕ, GAUSS' LAW reads (in spherical coordinates)

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{E}} = \frac{1}{r\sin\theta}\frac{\partial E}{\partial\phi} = 0\,. \quad \checkmark \tag{1}$$

$$\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{E}} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(E\sin\theta\right)\hat{\boldsymbol{r}} - \frac{1}{r}\frac{\partial}{\partial r}\left(rE\right)\hat{\boldsymbol{\theta}} = \frac{1}{r\sin\theta}\left(E\cos\theta + \sin\theta\frac{\partial E}{\partial\theta}\right)\hat{\boldsymbol{r}} - \frac{1}{r}\left(E + r\frac{\partial E}{\partial r}\right)\hat{\boldsymbol{\theta}}\,. \tag{2}$$

$$\text{Now,}\quad \frac{\partial E}{\partial\theta} = \frac{A\cos\theta}{r}\left[\cos u - \frac{1}{kr}\sin u\right] = E\,\frac{\cos\theta}{\sin\theta} \tag{3}$$

$$\begin{aligned}\text{and}\quad \frac{\partial E}{\partial r} &= -\frac{A\sin\theta}{r^2}\left(\cos u - \frac{\sin u}{kr}\right) + \frac{A\sin\theta}{r}\left(-k\sin u + \frac{\sin u}{kr^2} - \frac{k\cos u}{kr}\right)\\ &= \frac{A\sin\theta}{r^2}\left[-2\cos u + 2\frac{\sin u}{kr} - kr\sin u\right]\end{aligned} \tag{4}$$

$$\text{so}\quad \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{E}} = \frac{A}{r^2}\left\{2\cos\theta\left[\cos u - \frac{1}{kr}\sin u\right]\hat{\boldsymbol{r}} \;+\; \sin\theta\left[\cos u + \left(kr - \frac{1}{kr}\right)\sin u\right]\hat{\boldsymbol{\theta}}\right\}\,. \tag{5}$$

In order to satisfy FARADAY'S LAW we must therefore have (within a constant of integration)

$$\vec{\boldsymbol{B}} = -\int\left(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{E}}\right)dt = -\frac{A}{r^2}\left\{2\cos\theta\left[C - \frac{1}{kr}S\right]\hat{\boldsymbol{r}} + \sin\theta\left[C + \left(kr - \frac{1}{kr}\right)S\right]\hat{\boldsymbol{\theta}}\right\} \tag{6}$$

$$\text{where}\quad C \equiv \int\cos u\,dt = -\frac{\sin u}{\omega} \quad\text{and}\quad S \equiv \int\sin u\,dt = \frac{\cos u}{\omega}\,. \qquad (\text{Note: } \omega = ck\,.) \tag{7}$$

$$\text{Thus}\quad \vec{\boldsymbol{B}} = \frac{A}{ckr^2}\left\{2\cos\theta\left[\sin u + \frac{1}{kr}\cos u\right]\hat{\boldsymbol{r}} + \sin\theta\left[\sin u - \left(kr - \frac{1}{kr}\right)\cos u\right]\hat{\boldsymbol{\theta}}\right\} \tag{8}$$

or $\vec{\boldsymbol{B}} = B_r\,\hat{\boldsymbol{r}} + B_\theta\,\hat{\boldsymbol{\theta}}$ where

$$B_r = \frac{2A\cos\theta}{ckr^2}\left[\sin u + \frac{1}{kr}\cos u\right] \quad\text{and}\quad B_\theta = \frac{A\sin\theta}{ckr^2}\left[\sin u - \left(kr - \frac{1}{kr}\right)\cos u\right]\,. \tag{9}$$

This should satisfy GAUSS' LAW too: $\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{B}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 B_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta B_\theta\right)$

$$\begin{aligned} &= \frac{2A\cos\theta}{ckr^2}\frac{\partial}{\partial r}\left(\sin u + \frac{1}{kr}\cos u\right) + \frac{A}{ckr^3\sin\theta}\left[\sin u + \left(\frac{1}{kr} + kr\right)\cos u\right]\frac{\partial}{\partial\theta}\left(\sin^2\theta\right)\\ &= \frac{2A\cos\theta}{ckr^2}\left\{\left(k\cos u - \frac{1}{kr^2}\cos u - \frac{1}{r}\sin u\right) + \frac{1}{r}\left[\sin u - \left(kr - \frac{1}{kr}\right)\cos u\right]\right\}\\ &= \frac{2A\cos\theta}{ckr^2}\left[k\cos u - \frac{1}{kr^2}\cos u - \frac{1}{r}\sin u + \frac{1}{r}\sin u - k\cos u + \frac{1}{kr^2}\cos u\right] = 0\,. \quad\checkmark\end{aligned} \tag{10}$$

It remains only to check AMPÈRE'S LAW: $\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}} = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rB_\theta\right) - \frac{\partial B_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$ or

$$\begin{aligned}\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}} \;=\; \frac{1}{r}\Bigg\{&\frac{A\sin\theta}{ckr^2}\left[\sin u - \left(kr - \frac{1}{kr}\right)\cos u\right]\\ &+ \frac{A\sin\theta}{ckr}\left[k\cos u - \left(k + \frac{1}{kr^2}\right)\cos u + k\left(kr - \frac{1}{kr}\right)\sin u\right]\end{aligned}$$

$$
\begin{aligned}
& + \frac{2A\sin\theta}{ckr^2}\left[\sin u + \frac{\cos u}{kr}\right]\Big\}\,\hat{\boldsymbol{\phi}} \\
= & \frac{A\sin\theta}{ckr^3}\Big\{-\left[\sin u - \left(kr - \frac{1}{kr}\right)\cos u\right] \\
& + \left[kr\cos u - \left(kr + \frac{1}{kr}\right)\cos u + \left(k^2r^2 - 1\right)\sin u\right] \\
& + 2\left[\sin u + \frac{\cos u}{kr}\right]\Big\}\,\hat{\boldsymbol{\phi}} \\
\text{giving } \vec{\nabla}\times\vec{B} = & \frac{A\sin\theta}{cr^2}\left(\cos u + kr\sin u\right)\hat{\boldsymbol{\phi}}\,. \qquad (11)
\end{aligned}
$$

Now, if we're to get any joy from this, it had better be equal to $\dfrac{1}{c^2}\dfrac{\partial\vec{E}}{\partial t} = \dfrac{k^2}{\omega^2}\dfrac{\partial\vec{E}}{\partial t} = \dfrac{k}{c\omega}\dfrac{\partial\vec{E}}{\partial t}$

$$
\begin{aligned}
& = \frac{k}{c\omega}\frac{A\sin\theta}{r}\frac{\partial}{\partial t}\left[\cos u - \frac{\sin u}{kr}\right]\hat{\boldsymbol{\phi}} = \frac{A\sin\theta}{r}\frac{k}{c\omega}\left[\omega\sin u + \omega\frac{\cos u}{kr}\right]\hat{\boldsymbol{\phi}} \\
& = \frac{A\sin\theta}{cr^2}\left(kr\sin u + \cos u\right)\hat{\boldsymbol{\phi}}\,. \quad \checkmark\ \mathcal{QED} \qquad (12)
\end{aligned}
$$

Thus the proposed function does satisfy all of MAXWELL'S EQUATIONS as advertised and is therefore also a valid solution of TWE (The Wave Equation). And this is the simplest possible spherical wave! (Don't you just love curvilinear coordinates?)

(*b*) Calculate the Poynting vector. Average $\vec{S}$ over a full cycle to get the intensity vector $\vec{I}$. Does $\vec{I}$ point in the expected direction? Does it fall off like r^{-2}, as it should? **ANSWER**:

$$
\begin{aligned}
\vec{S} & = \frac{\vec{E}\times\vec{B}}{\mu_0} = \frac{1}{\mu_0}\left(E\,\hat{\boldsymbol{\phi}}\right)\times\left(B_r\,\hat{\boldsymbol{r}} + B_\theta\,\hat{\boldsymbol{\theta}}\right) = -\frac{1}{\mu_0}\left(EB_r\,\hat{\boldsymbol{\theta}} + EB_\theta\,\hat{\boldsymbol{r}}\right) \\
& = -\frac{A^2}{\mu_0 ckr^3}\Big[2\sin\theta\cos\theta\left(\cos u - \frac{\sin u}{kr}\right)\left(\sin u + \frac{\cos u}{kr}\right)\hat{\boldsymbol{\theta}} \\
& \quad + \sin^2\theta\left(\cos u - \frac{\sin u}{kr}\right)\left(\sin u - kr\cos u + \frac{\cos u}{kr}\right)\hat{\boldsymbol{r}}\Big] \\
& = -\frac{A^2}{\mu_0 ckr^3}\Big[2\sin\theta\cos\theta\left(\cos u\ \sin u + \frac{\cos^2 u}{kr} - \frac{\sin^2 u}{kr} - \frac{\sin u\ \cos u}{kr}\right)\hat{\boldsymbol{\theta}} \\
& \quad + \sin^2\theta\left(\cos u\ \sin u - kr\cos^2 u + \frac{\cos^2 u}{kr} - \frac{\sin^2 u}{kr} + \sin u\ \cos u - \frac{\sin u\ \cos u}{k^2r^2}\right)\hat{\boldsymbol{r}}\Big] \\
& = -\frac{A^2}{\mu_0 ckr^3}\Big\{2\sin\theta\cos\theta\left[\cos u\ \sin u\left(1 - \frac{1}{kr}\right) + \frac{\cos^2 u - \sin^2 u}{kr}\right]\hat{\boldsymbol{\theta}} \\
& \quad + \sin^2\theta\left[\cos u\ \sin u\left(2 - \frac{1}{k^2r^2}\right) - kr\cos^2 u + \frac{\cos^2 u - \sin^2 u}{kr}\right]\hat{\boldsymbol{r}}\Big\}\,. \qquad (13)
\end{aligned}
$$

The fact that $\vec{S}$ has a non-radial component may seem alarming, but let's check the time average: all of $\sin u\ \cos u$, $\sin^2 u$ and $\cos^2 u$ oscillate in time, but only the first averages to zero; the other two average to $\frac{1}{2}$, but their *difference* does average to zero. Thus

$$
\vec{I} \equiv \langle\vec{S}\rangle = \frac{A^2\sin^2\theta}{\mu_0 ckr^3}\,\frac{kr}{2}\,\hat{\boldsymbol{r}} = \frac{A^2\sin^2\theta}{2\mu_0 c}\,\frac{\hat{\boldsymbol{r}}}{r^2}\,, \qquad (14)
$$

which points radially outward and falls off like $1/r^2$, as expected.

(*c*) Integrate $\vec{I}\cdot d\vec{a}$ over a spherical surface to determine the total power radiated.

[You should get $P = 4\pi A^2/3\mu_0 c$.] **ANSWER**:

$$
P = \iint \vec{I}\cdot d\vec{a} = \frac{A^2}{2\mu_0 c}\int_0^\pi \frac{\sin^2\theta}{r^2}\,2\pi r^2\sin\theta d\theta = \frac{\pi A^2}{\mu_0 c}\int_0^\pi \sin^3\theta d\theta = -\frac{\pi A^2}{\mu_0 c}\int_1^{-1}\left(1-\cos^2\theta\right)d(\cos\theta)
$$

$$
\text{or}\quad P = \frac{\pi A^2}{\mu_0 c}\int_{-1}^{+1}(1-x^2)dx = \frac{\pi A^2}{\mu_0 c}\left(2 - \frac{2}{3}\right) \qquad \text{or}\quad \boxed{P = \frac{4\pi A^2}{3\mu_0 c}}\,. \quad \checkmark
$$

This was a tedious problem; it took me all day to get it right. I will be duly impressed if you managed to grind through it successfully. Now you know why we like our plane waves so much, Huygens' principle notwithstanding!