Diffraction as "∞-Slit Interference"

If I_0 is the **intensity** at the central maximum, $A_0 = \sqrt{(I_0)}$ is the **amplitude** at the central maximum. By dividing up the slit into N infinitesimal "pseudoslits", each contributing an amplitude $A_1 = A_0/N$, and letting $N \to \infty$, we get

$$A = A_0 \,(\sin \alpha)/\alpha \qquad \text{or} \qquad \boxed{I = I_0 \,[(\sin \alpha)/\alpha]^2}$$

where $\alpha \equiv \pi\,(a \sin \vartheta)/\lambda$ and a = the width of the slit.

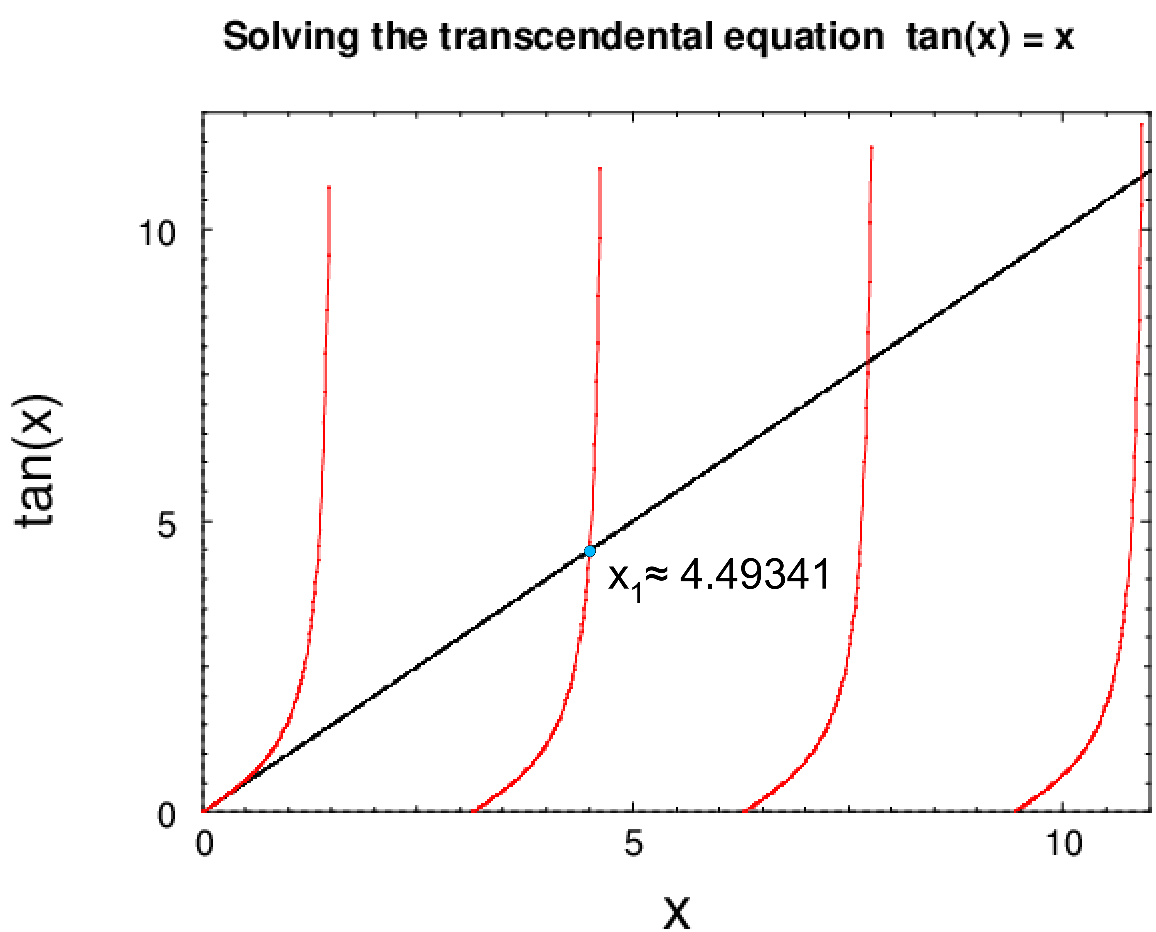
This **diffraction pattern** has its **first minimum** where $\alpha = \pi$ or

$$\boxed{a \sin \vartheta_1 = \lambda.}$$

The **maxima** of $I(\alpha)$ are found where the **slope** of $(\sin \alpha)/\alpha$ is **zero**: setting the derivative equal to zero gives (after some algebra) $\boxed{\tan \alpha = \alpha}$ which is a transcendental equation best solved **graphically**.

Maxima of Diffraction Pattern

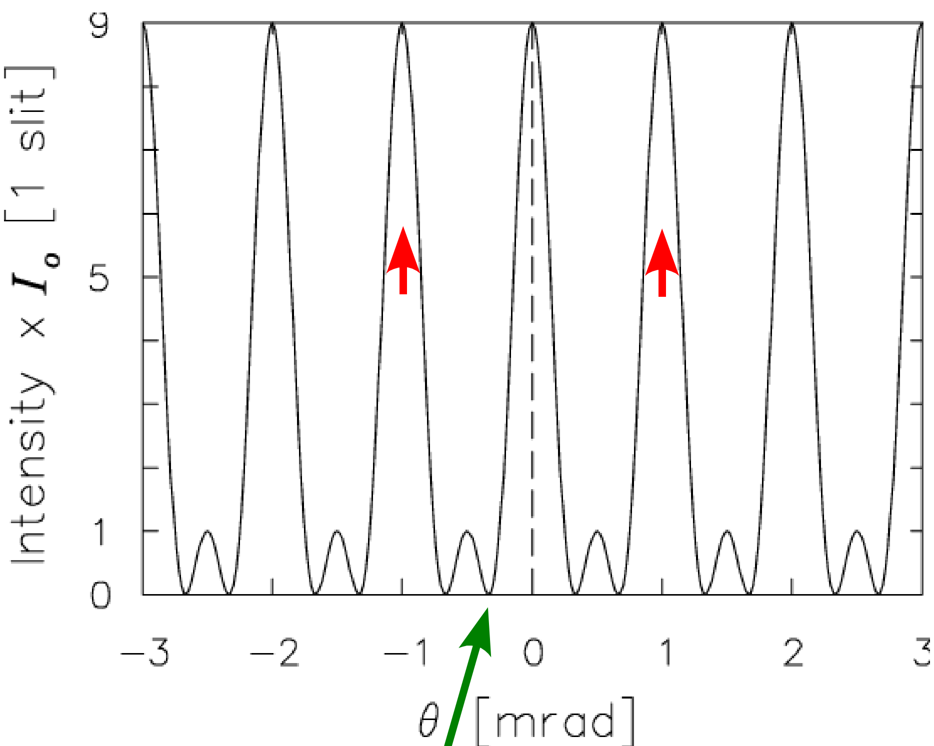
The **maxima** of $I(\alpha)$ occur where $\tan\alpha = \alpha$, a transcendental equation best solved **graphically:**



N-Slit Gratings

What's the difference? Gratings have **finite-width** (a) **slits!**

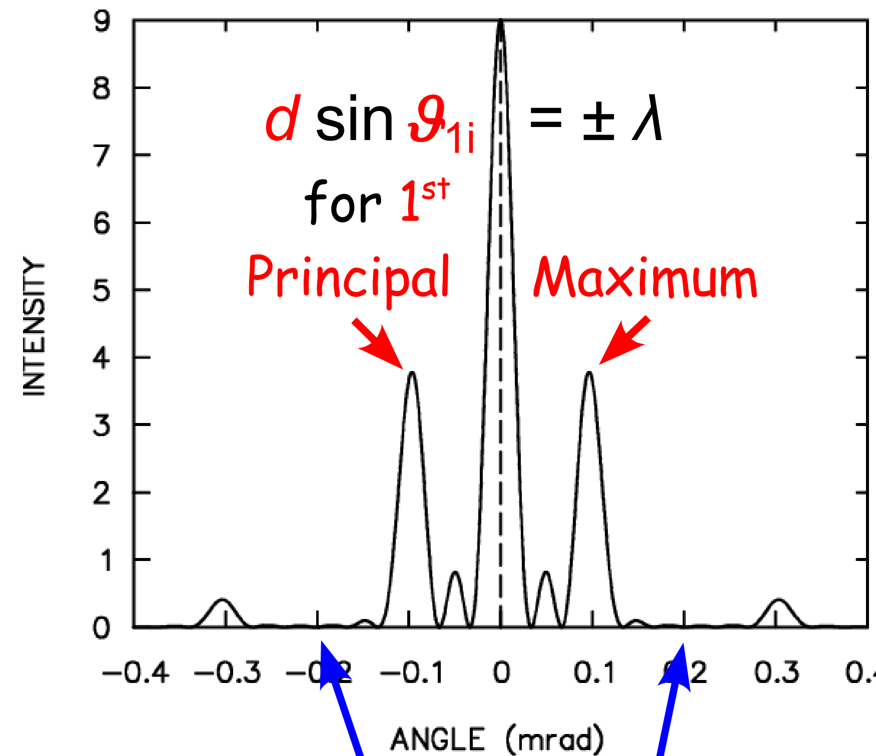
Ideal 3-slit intensity pattern



$d \sin \vartheta_{wid} = \pm \lambda / N$

for **width** of Principal Maxima.

3-slit Grating intensity pattern



$a \sin \vartheta_{1d} = \pm \lambda.$

for 1st diffraction minimum.

Be careful not to confuse ϑ_{1i} with ϑ_{1d} ! The formulae look alike but. . . .

Circular Apertures

We have been talking about "slits" as if all diffraction problems were one-dimensional. In reality, the most common type is circular, such as telescopes, laser cannons and the pupil of your eye. The following handwaving logic is not a proof, but a plausibility argument:

The narrower the slit, the wider the diffraction pattern. (Look at the formula for the first minimum!) Picture a circular aperture as a square aperture with the "corners chopped off": on average, it is narrower than the original square whose side was equal to the circle's diameter. Thus you would expect it to produce a wider diffraction pattern. Indeed it does! The numerical difference is a factor of 1.22: instead of $a \sin \vartheta_1 = \lambda$ we have

$$a \sin \vartheta_1 = 1.22\,\lambda\,.$$

Resolution

Diffraction is "reversible" in the sense that two light sources **inside** a telescope shining out through an aperture would produce overlapping patterns at a distant detector for the same conditions that two point sources at the **detector's** location would no longer be **resolvable** by the telescope's optics.

Thus the criterion for "**resolvability**" of two distant stars (for example) by a telescope of diameter a is that their angular separation be greater than the angle ϑ_{1d} given by $a \sin \vartheta_1 = 1.22\,\lambda$ between each one's central maximum and the first minimum of its diffraction pattern. This "**Rayleigh criterion**" also applies for microscopes, the pupil of the eye, *etc.*

Dispersion

Since the angular pattern of interference and diffraction from a grating explicitly depends on the wavelength λ, it follows that light of different wavelengths will be "bent" by different angles. If we consider the m^{th} Principal Maximum, for which $d \sin \vartheta_m = m\lambda$, and take the derivative of both sides with respect to λ, we get $d \cos \vartheta_m (\mathrm{d}\vartheta_m/\mathrm{d}\lambda) = m$ or

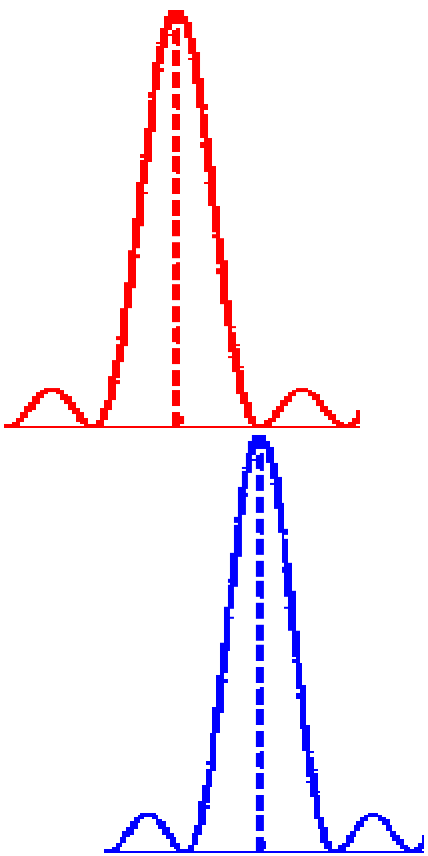
$\mathrm{d}\vartheta_m/\mathrm{d}\lambda = m/d \cos \vartheta_m$ *giving* $\boxed{\mathrm{d}\vartheta_m/\mathrm{d}\lambda \approx m/d}$ for small angles.

Thus if the purpose of our grating is to "resolve" different colours (known as "dispersion") then we want to have the smallest possible slit spacing d and the largest possible "order" m.

$\mathrm{d}\vartheta_m/\mathrm{d}\lambda \equiv D$ is known as "the Dispersion" of a grating.

Resolving Power

Again assuming we are using a grating to measure λ, <u>how close together</u> $(\Delta\lambda)$ <u>can two wavelengths be</u> $(\lambda' = \lambda + \Delta\lambda)$ <u>and still be resolved</u>? That is, if λ has an m^{th} order Principal Maximum (PM_m) at ϑ_m then λ' has **<u>its</u>** PM_m at $\vartheta_m' = \vartheta_m + \Delta\vartheta_m$ right on top of the <u>1^{st} minimum beyond</u> ϑ_m for λ .



We know that $d \sin \vartheta_m = m\lambda$, $d \sin \vartheta_m' = m\lambda'$ and $d \sin (\vartheta_m + \Delta\vartheta_m) = (m + 1/N)\,\lambda$ so that the extra path length difference between adjacent slits, λ/N, ensures a phasor diagram that closes on itself in N phasors. So we must have $m\lambda' = (m + 1/N)\,\lambda$ or $\lambda + \Delta\lambda = \lambda + \lambda/mN$ or

$$\lambda/\Delta\lambda = mN \equiv R,$$

the "Resolving power" of the grating.