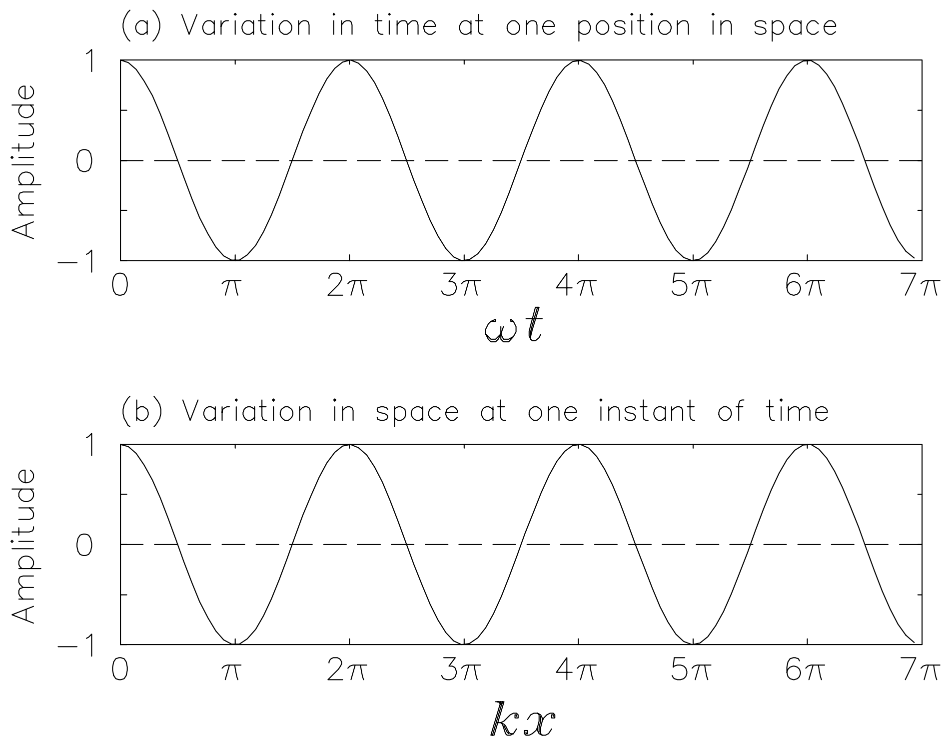
WAVES



Two views of a wave.

Traveling Waves

The wave *amplitude* A is a function of position $\vec{\boldsymbol{r}}$ and time t: $A(\vec{\boldsymbol{r}}, t)$. *At any fixed position* $\vec{\boldsymbol{r}}$, A *oscillates in time* at a frequency ω. We can describe this statement mathematically by saying that the entire *time dependence* of A is contained in [the real part of] a factor $e^{-i\omega t}$ (that is, the amplitude *at any fixed position* obeys *SHM*).*

*Note that $e^{+i\omega t}$ would have worked just as well, since the real part is the same as for $e^{-i\omega t}$. The choice of sign does matter, however, when we write down the *combined* time and space dependence in Eq. (4), which see.

The oscillation with respect to *position* $\vec{\boldsymbol{r}}$ *at any instant of time* t is given by the analogous factor $e^{i\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}}$ where $\vec{\boldsymbol{k}}$ is the *wave vector*; it points in the *direction of propagation* of the wave and has a magnitude (called the "wavenumber") k given by

$$k = \frac{2\pi}{\lambda} \tag{1}$$

where λ is the *wavelength*. Note the analogy between k and

$$\omega = \frac{2\pi}{T} \tag{2}$$

where T is the *period* of the oscillation in time at a given point. You should think of λ as the "period in space."

We may simplify the above description by *choosing our coordinate system* so that the x axis is *in the direction of* $\vec{\boldsymbol{k}}$, so that[†] $\vec{\boldsymbol{k}} \cdot \vec{\boldsymbol{r}} = k\,x$. Then the amplitude A no longer depends on y or z, only on x and t.

We are now ready to give a full description of the function describing this wave:

$$A(x,t) \;=\; A_0\, e^{ikx} \cdot e^{-i\omega t}$$

or, recalling the multiplicative property of the exponential function, $e^a \cdot e^b = e^{(a+b)}$,

$$A(x,t) \;=\; A_0\, e^{i(kx-\omega t)}. \tag{3}$$

[†]In general $\vec{\boldsymbol{k}} \cdot \vec{\boldsymbol{r}} = xk_x + yk_y + zk_z$. If $\vec{\boldsymbol{k}} = k\,\hat{\boldsymbol{\imath}}$ then $k_x = k$ and $k_y = k_z = 0$, giving $\vec{\boldsymbol{k}} \cdot \vec{\boldsymbol{r}} = k\,x$.

To achieve complete generality we can restore the vector version:

$$A(x,t) \; = \; A_0 \; e^{i\left(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}-\omega t\right)} \tag{4}$$

This is the preferred form for a general description of a **plane wave**, but for present purposes the scalar version (3) suffices. Using Eqs. (1) and (2) we can also write the plane wave function in the form

$$A(x,t) \; = \; A_0 \; \exp\left[2\pi i\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right] \tag{5}$$

but you should strive to become completely comfortable with k and ω — we will be seeing a lot of them in Physics!

Speed of Propagation

Neither of the images in Fig. 1 captures the most important qualitative feature of the wave: namely, that it *propagates* — *i.e.* moves steadily along in the direction of $\vec{k}$. If we were to let the *snapshot* in Fig. 1b become a *movie*, so that the time dependence could be seen vividly, what we would see would be the same wave pattern *sliding along the graph to the right* at a steady rate. *What* rate? Well, the answer is most easily given in simple qualitative terms:

The wave has a distance λ (one *wavelength*) between "crests." Every *period* T, one full wavelength passes a fixed position. Therefore a given crest travels a distance λ in a time T so the *velocity of propagation* of the wave is just

$$c = \frac{\lambda}{T} \qquad \text{or} \qquad c = \frac{\omega}{k} \tag{6}$$

where I have used c as the symbol for the propagation velocity even though this is a completely *general* relationship between the frequency ω, the wave vector magnitude k and the propagation velocity c of *any* sort of wave, not just electromagnetic waves (for which c has its most familiar meaning, namely the speed of light).

This result can be obtained more easily by noting that A is a function *only* of the *phase* θ of the oscillation,

$$\theta \equiv kx - \omega t \tag{7}$$

and that the criterion for "seeing the same waveform" is $\theta =$ constant or $d\theta = 0$. If we take the differential of Eq. (7) and set it equal to zero, we get

$$d\theta = k\,dx - \omega\,dt = 0 \quad \text{or} \quad k\,dx = \omega\,dt$$

$$\text{or} \quad \frac{dx}{dt} = \frac{\omega}{k}.$$

But $dx/dt = c$, the propagation velocity of the waveform. Thus we reproduce Eq. (6).

This treatment also shows why we chose $e^{-i\omega t}$ for the time dependence so that Eq. (7) would describe the phase: if we used $e^{+i\omega t}$ then the phase would be $\theta \equiv kx + \omega t$ which gives $dx/dt = -c$, — *i.e.* a waveform propagating in the negative x direction (to the *left* as drawn).

Since $(kx - \omega t) = k(x - ct)$, Eq. (4) can be written

$$A(x,t) \;=\; A_0 \, e^{ik(x-ct)}$$

and we can extend the above argument to waveforms that are not of the ideal sinusoidal shape shown in Fig. 1; in fact it is more vivid if one imagines some special shape like (for instance) a *pulse* propagating down a string at velocity c. As long as $A(x,t)$ *is a function only of* $x' = x - ct$, *no matter what its shape*, it will be *static in time* when viewed by an observer traveling along with the wave[‡] at velocity c. This doesn't require any elaborate derivation; x' is just the position measured in such an observer's reference frame!

‡Don't try this with an electromagnetic wave! The argument shown here is explicitly *nonrelativistic*, although a more mathematical proof reaches the same conclusion without such restrictions.

The Wave Equation

Suppose we know that we have a *traveling wave* $A(x,t) = A_0 \cos(kx - \omega t)$.

At a *fixed position* $(x = \text{const})$ we see *SHM* in time:

$$\left(\frac{\partial^2 A}{\partial t^2}\right)_x = -\omega^2 A \qquad (8)$$

(Read: "The second partial derivative of A with respect to time [*i.e.* the *acceleration* of A] with x held fixed is equal to $-\omega^2$ times A itself.") *I.e.* we must have a *linear restoring force*.

Similarly, if we take a "snapshot" (hold t fixed) and look at the *spatial* variation of A, we find the oscillatory behaviour analogous to *SHM*,

$$\left(\frac{\partial^2 A}{\partial x^2}\right)_t = -\,k^2\,A \tag{9}$$

(Read: "The second partial derivative of A with respect to position [*i.e.* the *curvature* of A] with t held fixed is equal to $-k^2$ times A itself.")

Thus

$$A = -\frac{1}{\omega^2}\left(\frac{\partial^2 A}{\partial t^2}\right)_x = -\frac{1}{k^2}\left(\frac{\partial^2 A}{\partial x^2}\right)_t .$$

If we multiply both sides by $-k^2$, we get

$$\frac{k^2}{\omega^2}\left(\frac{\partial^2 A}{\partial t^2}\right)_x = \left(\frac{\partial^2 A}{\partial x^2}\right)_t .$$

But $\omega = ck$ so $\dfrac{k^2}{\omega^2} = \dfrac{1}{c^2}$, giving

the **Wave Equation**

$$\boxed{\frac{\partial^2 A}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = 0} \qquad (10)$$

In words, the *curvature* of A is equal to $1/c^2$ times the *acceleration* of A at any (x, t) point.

Whenever you see this differential equation governing some quantity A, *i.e.* where the acceleration of A is proportional to its curvature, you know that $A(x,t)$ will exhibit wave motion!

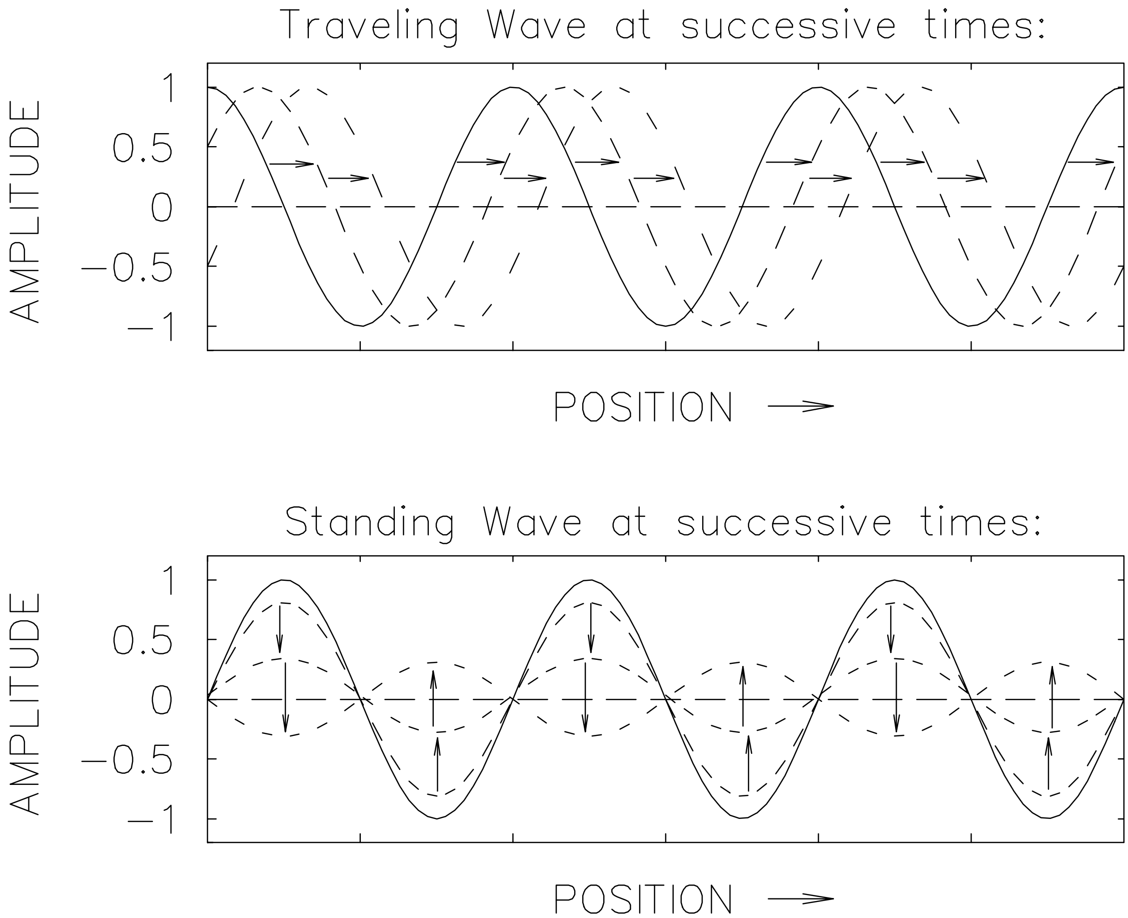
Linear Superposition

As long as the *acceleration* is strictly proportional to the *curvature* we have an important consequence: *several different waves can propagate independently through the same medium*. The displacement at any given time and place is just the *linear sum* of the displacements due to each of the simultaneously propagating waves. This is known as the **principle of linear superposition**, and it is essential to our understanding of wave phenomena.

In general the overall displacement $A(x,t)$ resulting from the linear superposition of two waves $A_1 e^{i(k_1 x - \omega_1 t)}$ and $A_2 e^{i(k_2 x - \omega_2 t)}$ is given by

$$A(x,t) = A_1 e^{i(k_1 x - \omega_1 t)} + A_2 e^{i(k_2 x - \omega_2 t)}. \qquad (11)$$

Let's look at a few simple examples.



Traveling *vs.* standing waves.

Standing Waves

Consider the case where $A_1 = A_2 = A_0$, $k_1 = k_2 = k$ and $\omega_1 = -\omega_2 = \omega$. That is, two otherwise identical waves *propagating in opposite directions*. The algebra is simple:

$$\begin{aligned} A(x,t) &= A_0 \left[e^{i(kx-\omega t)} + e^{i(kx+\omega t)} \right] \\ &= A_0 e^{ikx} \left[e^{-i\omega t} + e^{+i\omega t} \right] \\ &= A_0 e^{ikx} [\cos(\omega t) - i \sin(\omega t) \\ &\qquad + \cos(\omega t) + i \sin(\omega t)] \\ &= 2A_0 \cos(\omega t) e^{ikx}. \end{aligned} \tag{12}$$

The real part of this (which is all we ever actually use) describes a sinusoidal waveform of wavelength $\lambda = 2\pi/k$ whose *amplitude* $2A_0 \cos(\omega t)$ oscillates in time but which does *not* propagate in the x direction.

Standing waves are very common, especially where a traveling wave is *reflected* from a boundary, since this automatically creates a second wave of similar amplitude and wavelength propagating back in the opposite direction — the very condition assumed at the beginning of this discussion.

Water Waves

Although all sorts of waves are ubiquitous in our lives,[§] our most familiar "wave experiences" are probably with *water waves*, which are unfortunately one of the *least simple* types of waves. Although water waves are routinely used for illustration, they are rarely discussed in great depth (heh, heh) in any introductory Physics texts. They do, however, serve to illustrate one important feature of waves, namely that *not all waves obey* the simple relationship $c = \omega/k$ for their *propagation velocity* c.

[§]Indeed, we are *made* of waves, as **quantum mechanics** has taught us!

Let's restrict ourselves to *deep ocean* waves, where the "restoring force" is simply *gravity*. (When a wave reaches shallow water, the bottom provides an immobile boundary that complicates matters severely, as anyone knows who has ever watched surf breaking on a beach!) The motion of an "element" of water in such a wave is *not* simply "up and down" as we pretended at the beginning of this chapter, but a *superposition* of "up and down" with "back and forth" in the direction of wave propagation. A cork floating on the surface of such a wave executes *circular* motion, or so I am told.

More importantly, the *propagation velocity* of such waves is *higher* for *longer wavelength*.

Phase *vs.* Group Velocity

The precise relationship between angular frequency ω and wavenumber k for deep-water waves is

$$\omega = \sqrt{\frac{g\,k}{2}} \tag{13}$$

where g has its usual meaning. Such a functional relationship $\omega(k)$ between frequency and wavenumber is known as the **dispersion relation** for waves in that medium.

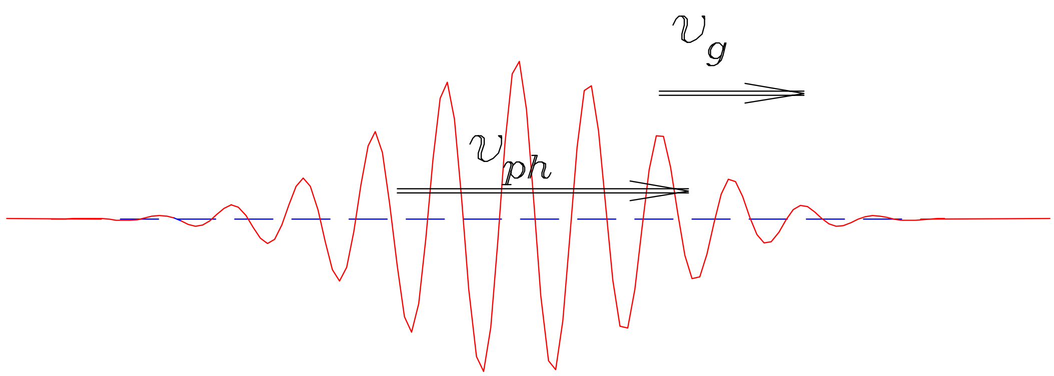
If we have a simple traveling plane wave $A(x,t) = A_0 \exp[i(kx - \omega t)]$, with no beginning and no end, the rate of propagation of a point of constant phase (known as the **phase velocity** $v_{\rm ph}$) is still given by

$$\boxed{v_{\rm ph} \equiv \frac{\omega}{k}} \tag{14}$$

However, by combining Eq. (14) with Eq. (13) we find that the phase velocity is *higher* for *smaller* k (longer λ):

$$v_{\rm ph} = \sqrt{\frac{g}{2k}}\,. \tag{15}$$

Moreover, such a wave *carries no information*. It has been passing by forever and will continue to do so forever; it is the same amplitude everywhere; and so on. Obviously our **plane wave** is a bit of an over-simplification. If we want to send a *signal* with a wave, we have to turn it on and off in some pattern; we have to make wave *pulses* (or, anticipating the terminology of **quantum mechanics**, "**wave packets**"). And when we do that with water waves, we notice something odd: *the wave packets propagate slower than the "wavelets" in them!*



Such a packet is a superposition of waves with different wavelengths; the k-dependence of v_{ph} causes a phenomenon known as **dispersion**, in which waves of different wavelength, initially moving together in phase, will drift apart as the packet propagates, making it "broader" in both space and time. (Obviously such a **dispersive medium** is undesirable for the transmission of information!) But how do we determine the effective speed of transmission of said information — *i.e.* the propagation velocity *of the packet itself*, called the **group velocity** v_{g}?

The *general* definition of the group velocity (the speed of transmission of information and/or energy in a wave packet) is

$$\boxed{\, v_{\rm g} \;\equiv\; \frac{\partial\omega}{\partial k} \,} \;. \qquad (16)$$

For the particular case of deep-water waves, Eq. (16) combined with Eq. (13) gives

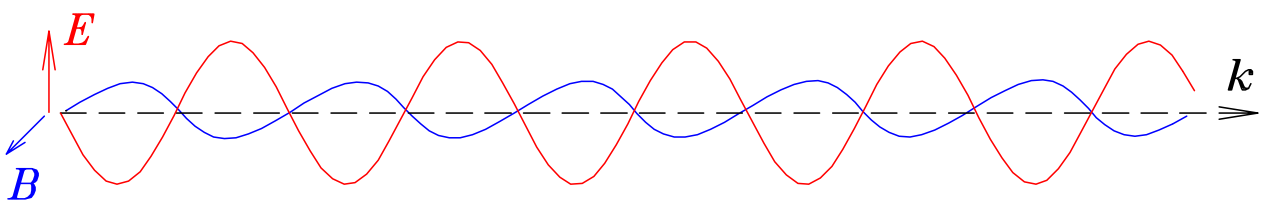
$$v_{\rm g} \;=\; \frac{1}{2}\sqrt{\frac{g}{2k}}. \qquad (17)$$

That is, the *packet* propagates at *half* the speed of the "wavelets" within it. This behaviour can actually be observed in the wake of a large vessel on the ocean, seen from high above (*e.g.* from an airliner).

Such exotic-seeming wave phenomena are ubiquitous in all dispersive media, which are anything but rare. For now, however, we will restrict ourselves to waves propagating through simple *non*-dispersive media, for which the **dispersion relation** is just $\omega = c\,k$ with c constant, for which $v_{\mathsf{ph}} = v_{\mathsf{g}} = c$.

Electromagnetic Waves

We have some difficulty visualizing a *wave* consisting only of electric and magnetic *fields*. However, if we plot the strength of $\vec{\boldsymbol{E}}$ along one axis and the strength of $\vec{\boldsymbol{B}}$ along another (perpendicular) axis, then the direction of propagation $\widehat{\boldsymbol{k}}$ will be perpendicular to both $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$, as shown.

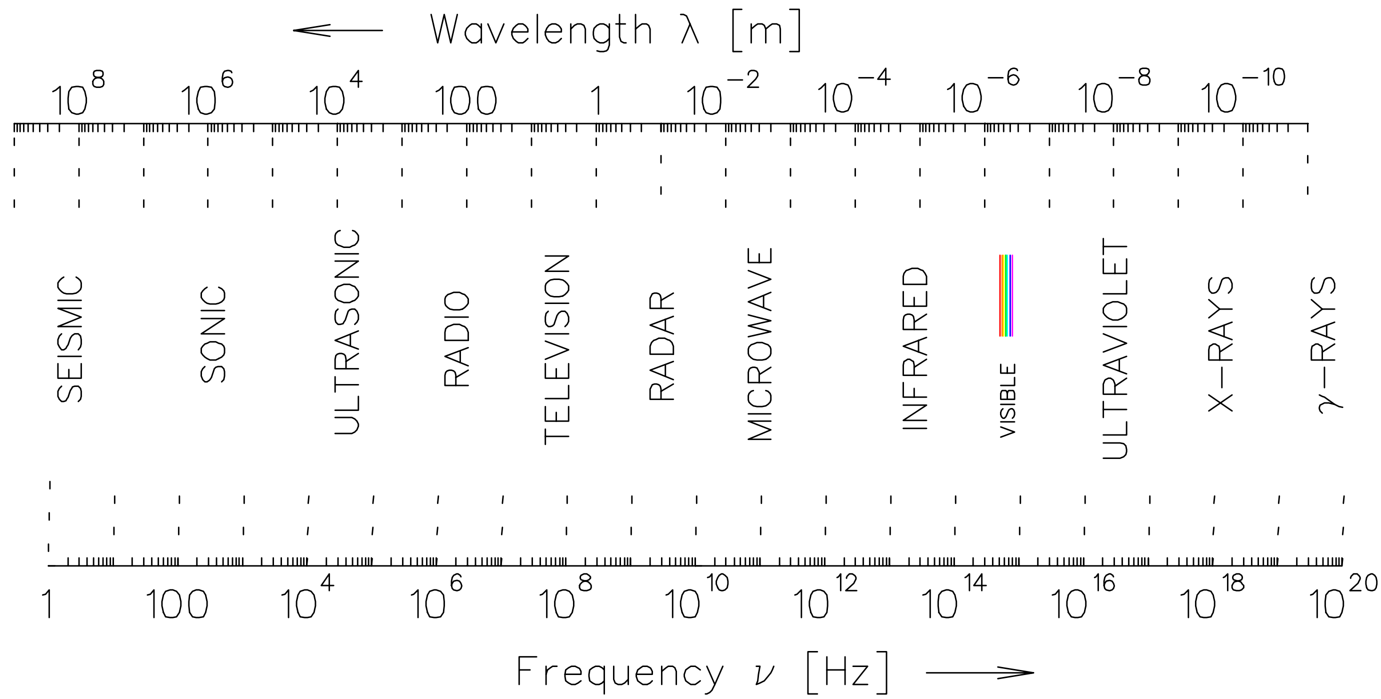


Polarization

The case shown is *linearly polarized*, which means simply that the $\vec{E}$ and $\vec{B}$ fields are in specific fixed directions. Of course, the directions of $\vec{E}$ and $\vec{B}$ could be interchanged, giving the "opposite" polarization. Polaroid sunglasses transmit the light waves with $\vec{E}$ vertical (which are not reflected efficiently off horizontal surfaces) and absorb the light waves with $\vec{E}$ horizontal (which are), thus reducing "glare" (reflected light from horizontal surfaces) without blocking out all light.

There is another possibility, namely that the two linear polarizations be *superimposed* so that both the $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ vectors *rotate* around the direction of propagation $\widehat{\boldsymbol{k}}$, remaining always perpendicular to $\widehat{\boldsymbol{k}}$ and to each other. This is known as *circular polarization*. It too comes in two versions, *right* circular polarization and *left* circular polarization, referring to the hand whose fingers curl in the direction of the rotation if the thumb points along $\widehat{\boldsymbol{k}}$.

The Electromagnetic Spectrum



Note logarithmic wavelength and frequency scales.

We have special names for electromagnetic (EM) waves of different wavelengths λ and frequencies ω.[¶] We call EM waves with $\lambda \gtrsim 1$ m "radio waves". If λ fits into a small appliance it is "microwave". At the short λ end of the microwave spectrum, "infrared" begins; the name is meant to suggest ω *below* the *red* end of the *visible light* spectrum of EM waves, which extends from $\lambda \sim 500$ nm (5000 Å) for red light through orange, yellow, green and blue to $\lambda \sim 200$ nm (2000 Å) for violet light. Beyond that we lose sight of the shorter wavelengths (so to speak) and the next range is called "ultraviolet", which fades into "x-rays" and finally "gamma rays" as ω increases and λ gets shorter.

[¶]If λ *in*creases (so that the wavenumber $k = 2\pi/\lambda$ *de*creases), then the frequency ω must *de*crease to match, since the ratio ω/k must always be equal to the same propagation velocity c.

Spherical Waves

The utility of thinking of $\vec{\boldsymbol{k}}$ as a "ray" becomes more obvious when we get away from plane waves and start thinking of waves with *curved* wavefronts. The simplest such wave is the type that is emitted when a pebble is tossed into a still pool — an example of the "point source" that radiates waves isotropically in all directions. The wavefronts are then *circles* in two dimensions (the surface of the pool) or *spheres* in three dimensions (as for sound waves) separated by one wavelength λ and heading outward from the source at the propagation velocity c.

In this case the "rays" k point along the radius vector $\hat{\boldsymbol{r}}$ from the source at any position and we can write down a simple formula for the "wave function" (displacement A as a function of position) that depends only on the time t and the *scalar* distance r from the source.

A plausible first guess would be just $A(x,t) = A_0 \; e^{i(kr-\omega t)}$, but this cannot be right! Why not? Because it violates energy conservation. The energy density stored in a wave is proportional to the square of its amplitude; in the trial solution above, the amplitude of the outgoing spherical wavefront is constant as a function or r, but the *area* of that wavefront increases as r^2. Thus the energy in the wavefront increases as r^2? I think not!

We can get rid of this effect by just dividing the amplitude by r (which divides the energy density by r^2). Thus a trial solution is

$$\boxed{A(x,t) \;=\; A_0 \, \frac{e^{i(kr-\omega t)}}{r}.} \tag{18}$$

which is, as usual, correct. ‖

‖The factor of $1/r$ accounts for the conservation of energy in the outgoing wave: since the spherical "wave front" distributes the wave's energy over a surface area $4\pi r^2$ and the flux of energy per unit area through a spherical surface of radius r is proportional to the *square* of the wave amplitude at that radius, the integral of $|f|^2$ over the entire sphere (*i.e.* the total outgoing *power*) is independent of r, as it must be.

We won't use this equation for anything right now, but it is interesting to know that it does accurately describe an outgoing** spherical wave.

**One can also have "incoming" spherical waves, for which Eq. (18) becomes

$$A(x,t) \;=\; A_0\,\frac{e^{i(kr+\omega t)}}{r}.$$