AC CIRCUITS

The term “AC” stands for “Alternating Current”, typically the 60 Hz power available from any North American electrical outlet.[1]

21.1 The Differential Equation

We begin by picturing a generic series-LCR circuit driven by a sinusoidal voltage $\mathcal{E}(t) = \mathcal{E}_0 \cos(\omega t) = \Re\, e^{i\omega t}$. It is convenient to use the complex form[2] for calculations; just remember that none of the actual physical quantities like current or voltage will actually have a measurable imaginary part.[3] The voltage amplitude $\mathcal{E}_0$ is taken to be pure real.

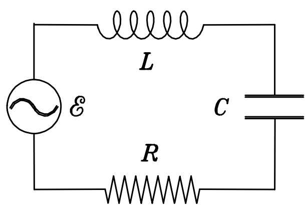

Figure 21.1 An LCR circuit driven by an AC voltage.

Applying Kirchhoff's rule of single-valued potential around this loop, we have

$$\mathcal{E} - L\ddot{Q} - \frac{1}{C}Q - R\dot{Q} = 0 \; . \tag{1}$$

When the AC power supply is first turned on, we generally have a very complicated behaviour involving the resonant (or “natural”) frequency

$$\omega_0 \equiv \frac{1}{\sqrt{LC}} \tag{2}$$

and the damping rate

$$\gamma \equiv \frac{R}{2L} = \frac{1}{2\tau_{RL}} \; , \tag{3}$$

just like the damped mass on a spring when an oscillatory driving mechanism is applied. If there were no resistor in the circuit, this “ringing” would go on indefinitely; but with the damping caused by R it eventually dies away and the circuit settles down to the only plausible “steady-state” motion, namely for Q to oscillate at the same frequency as the driving voltage:

$$Q(t) = Q_0 e^{i\omega t} \; . \tag{4}$$

Bearing in mind that the constant amplitude Q_0 may not be entirely real, let's see if this trial solution (4) “works” — *i.e.* satisfies the differential equation. One motive for using the complex exponential form is that it is so easy to take derivatives: each time derivative of $Q(t)$ just “pulls down” another factor of $i\omega$. Thus

$$\mathcal{E}_0 e^{i\omega t} + \omega^2 L Q_0 e^{i\omega t} - \frac{1}{C} Q_0 e^{i\omega t} - i\omega R Q_0 e^{i\omega t} = 0 \; , \tag{5}$$

from which we can remove the common factor $e^{i\omega t}$ and do a little algebra to obtain

$$Q_0 = \frac{\mathcal{E}_0/L}{\frac{1}{LC} - \omega^2 + i\frac{R}{L}\omega} \tag{6}$$

or [recalling the definitions (2) and (3)]

$$Q_0 = \frac{\mathcal{E}_0/L}{\omega_0^2 - \omega^2 + 2i\gamma\omega} \; . \tag{7}$$

Now, the charge on a capacitor cannot be measured directly; what we usually want to know is the *current* $I \equiv \dot{Q}$. Since the entire time dependence of Q is in the factor $e^{i\omega t}$, we have trivially

$$I(t) = i\omega Q(t) = I_0 e^{i\omega t} \tag{8}$$

where

$$I_0 = i\omega Q_0 = \frac{i\omega \mathcal{E}_0/L}{\omega_0^2 - \omega^2 + 2i\gamma\omega} \; . \tag{9}$$

Since everything we might want to know ($\mathcal{E}$, Q and I) has the same time dependence except for differences of *phase* encoded in the complex amplitudes Q_0 and I_0, we can think in terms of an *effective resistance* R_{eff} such that

$$\mathcal{E} = I R_{\text{eff}} \qquad \text{or} \qquad R_{\text{eff}} = \frac{\mathcal{E}_0}{I_0} \; . \tag{10}$$

With a little more algebra we can write the effective resistance in the form

$$R_{\text{eff}} = R - iX_C + iX_L \tag{11}$$

where

$$X_C \equiv \frac{1}{\omega C} \qquad \text{and} \qquad X_L \equiv \omega L \tag{12}$$

are respectively the *capacitive reactance* and the *inductive reactance* of the circuit. These are quantities that “act like” (and have the units of) *resistances* — just like R, the first term in R_{eff}.

The *current* through the circuit cannot be different in different places (due to charge conservation) and

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[1] In Europe the standard is 50 Hz.

[2] Here $\Re$ signifies “the real part of” a complex quantity like $e^{i\theta} = \cos\theta + i\sin\theta$. The imaginary part is written (*e.g.*) $\Im\, e^{i\theta} = \sin\theta$.

[3] Let me know if you invent an imaginary voltmeter!

follows the time dependence of the driving voltage but (because $R_{\rm eff}$ is generally complex) is not generally *in phase* with it, nor with *voltage drops* across C and L:

$$\begin{aligned} -\Delta\mathcal{E}_R &= IR\,, \quad \text{but} \\ -\Delta\mathcal{E}_C &= -iIX_C \quad \text{and} \\ -\Delta\mathcal{E}_L &= +iIX_L\,. \end{aligned} \tag{13}$$

From Eqs. (11) and (13) one can easily deduce the *phase differences* between these voltages at any time (for example, $t = \pi/\omega$) when $\mathcal{E}$ has its maximum negative real value: the voltage drop across R will be real and positive (it is always exactly out of phase with the driving voltage) but the voltage drop across the inductance will be in the positive imaginary direction — *i.e.* its real part will be zero at that instant, as will that of the voltage drop across the capacitor, which is then in the negative imaginary direction.

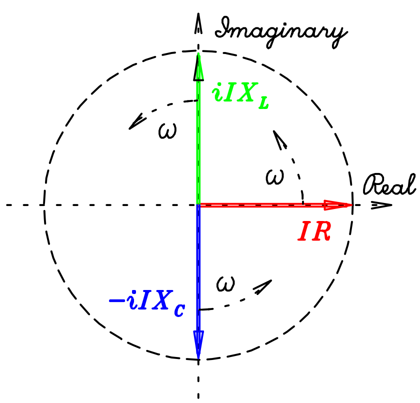

Figure 21.2 The "Phase Circle".

A convenient way of looking at this is with the "Phase Circle" shown in Fig. 21.2, where the "*directions*" of the voltage drops in "complex phase space" are shown as vectors. All three voltage drops "rotate" in this "phase space" at a constant frequency ω but their phase relationship is always preserved: namely, the voltage across the *capacitor* lags that across the resistor by an angle of $\pi/2$ and the voltage across the *inductance* leads that across the resistor by an angle of $\pi/2$.[4] At any other instant the real, measurable value of any of these voltages is just its real part, *i.e.* the projection of its complex vector onto the real axis.

21.2 Power

From the point of view of the power supply,[5] the circuit is a "black box" that "resists" the applied voltage with a rather weird "back $\mathcal{EMF}$" $\mathcal{E}_{\rm back}$ given by $R_{\rm eff}$ times the current I; $\mathcal{E}_{\rm back}$ is given by the sum of all three terms in Eq. (13) or the sum of the three vectors in Fig. 21.2. The *power* dissipated in the circuit is the product of the real part of the applied voltage[6] and the real part of the resultant current[7]

$$\begin{aligned} P(t) &= \Re\mathcal{E} \times \Re I = \Re\left(\mathcal{E}_0 e^{i\omega t}\right) \Re\left(I_0 e^{i\omega t}\right) \\ &= \mathcal{E}_0^2 \Re\left(\frac{1}{R_{\rm eff}}\right) \cos^2(\omega t)\,. \end{aligned} \tag{14}$$

which oscillates at a frequency 2ω between zero and its maximum value

$$P_{\rm max} = \mathcal{E}_0^2 \Re\left(\frac{1}{R_{\rm eff}}\right) \tag{15}$$

so that the *average* power drain is[8]

$$\langle P \rangle = \frac{1}{2}\mathcal{E}_0^2 \left[\frac{R}{R^2 + (X_L - X_C)^2}\right]\,. \tag{16}$$

A little more algebra will yield the practical formula

$$\langle P \rangle = \mathcal{E}_{\rm rms} I_{\rm rms} \cos\phi \tag{17}$$

where $\mathcal{E}_{\rm rms} = \mathcal{E}_0/\sqrt{2}$, $I_{\rm rms}$ is the root-mean-square current in the circuit,

$$\cos\phi = \frac{R}{Z} \tag{18}$$

is the "*power factor*" of the RC circuit and

$$Z \equiv \sqrt{R^2 + (X_L - X_C)^2} \tag{19}$$

is the *impedance* of the circuit.[9]

[4] There are many ways of remembering this phase relationship; I am not particularly fond of "$\mathcal{ELI}$ the $\mathcal{ICE}$ Man" because it refers only to the current and voltage in individual circuit elements and it has no explanatory aspect whatsoever. I prefer to think of it this way: when the current starts flowing there is immediately a voltage drop across the resistor, but it takes a while to charge up the capacitor, so it *lags behind*; the inductance, on the other hand, "fights" the establishment of a current in the first place, so it is *ahead of* the current. Use whatever works for you.

[5] Please forgive my anthropomorphization of circuit elements; these metaphors help me remember their "behaviour".

[6] The imaginary voltage component doesn't generate any power.

[7] Neither does the imaginary part of the current.

[8] I have used $\dfrac{1}{x+iy} = \dfrac{x-iy}{x^2+y^2}$ to obtain the real part of $1/R_{\rm eff}$.

[9] Expressing the average power dissipation in this form allows one to think of an AC circuit the same way as a DC circuit with the *power factor* as a sort of "fudge factor".