PHASORS

(Advanced Topic)

What happens when coherent light comes through more than two slits, all equally spaced a distance d apart, in a line parallel to the incoming wave fronts? The same criterion still holds for completely *con*structive interference (what we will now refer to as the PRINCIPAL MAXIMA) but we no longer have a simple criterion for *de*structive interference: each successive slit's contribution cancels out that of the adjacent slit, but if there are an *odd number of slits*, there is still one left over and the combined amplitude is not zero.

Does this mean there are *no* angles where the intensity goes to zero? Not at all; but it is not quite so simple to locate them. One way of making this calculation easier to visualize (albeit in a rather abstract way) is with the geometrical aid of PHASORS: A single wave

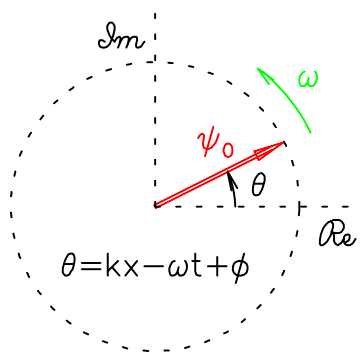

Figure 1 A single "PHASOR" of length ψ_0 (the wave amplitude) precessing at a frequency ω in the complex plane.

can be expressed as $\psi(x,t) = \psi_0 e^{i\theta}$ where $\theta = kx - \omega t + \phi$ is the *phase* of the wave at a fixed position x at a given time t. (As usual, ϕ is the "initial" phase at $x = 0$ and $t = 0$. At this stage it is usually ignored; I just retained it one last time for completeness.) If we focus our attention on one particular location in space, this single wave's "displacement" ψ at that location can be represented geometrically as a vector of length ψ_0 (the wave amplitude) in the complex plane called a "PHASOR" As time passes, the "direction" of the phasor rotates at an angular frequency ω in that abstract plane.

There is not much advantage to this geometrical description for a *single* wave (except perhaps that it engages the right hemisphere of the brain a little more than the algebraic expression) but when one goes to "add together" two or more waves with *different phases*, it helps a lot! For example, two waves of equal

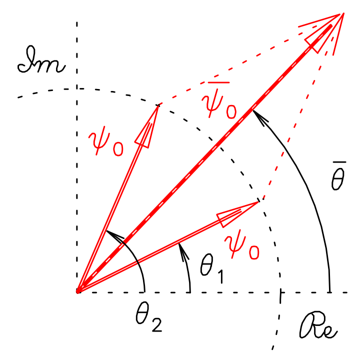

Figure 2 Two waves of equal amplitude ψ_0 but different phases θ_1 and θ_2 are represented as PHASORS in the complex plane. Their vector sum has the resultant amplitude $\bar{\psi}_0$ and the average phase $\bar{\theta}$.

amplitude but different phases can be added together algebraically as for BEATS:

$$\begin{aligned} \bar{\psi} &= \psi_0 \left[e^{i\theta_1} + e^{i\theta_2} \right] \\ &= 2\,\psi_0\, e^{i\bar{\theta}} \cos(\delta/2) \\ &= \bar{\psi}_0\, e^{i\bar{\theta}} \end{aligned} \tag{1}$$

where

$$\begin{aligned} \bar{\psi}_0 &= 2\,\psi_0\, \cos(\delta/2) \\ \bar{\theta} &\equiv \tfrac{1}{2}(\theta_1 + \theta_2) \\ \delta &\equiv \theta_2 - \theta_1\ . \end{aligned} \tag{2}$$

That is, the combined amplitude $\bar{\psi}_0$ can be obtained by adding the phasors "tip-to-tail" like ordinary vectors. Like the original components, the whole thing continues to precess in the complex plane at the common frequency ω.

We are now ready to use PHASORS to find the amplitude of an arbitrary number of waves of arbitrary amplitudes and phases but a common frequency and wavelength interfering at a given position. This is illustrated in Fig. 3 for 5 phasors. In practice, we rarely attempt

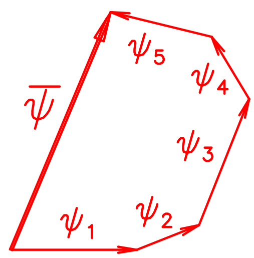

Figure 3 The net amplitude of a wave produced by the interference of an arbitrary number of other waves of the same frequency of arbitrary amplitudes ψ_j and phases θ_j can in principle be calculated geometrically by "tip-to-tail" vector addition of the individual PHASORS in the complex plane.

such an arbitrary calculation, since it cannot be simplified algebraically.

Instead, we concentrate on simple combinations of waves of equal amplitude with well defined phase differences, such as those produced by a regular array of parallel slits with an equal spacing between adjacent slits.

It will be conceptually helpful to show a geometrical explanation of the 6-slit interference pattern in Fig. 6 in terms of phasor diagrams, but clearly the smooth curve shown there is not the result of an infinte number of geometrical constructions. It comes from an algebraic formula that we can derive for an arbitrary angle ϑ and a corresponding phase difference $\delta = (2\pi d/\lambda)\sin\vartheta$ between rays from adjacent slits. The formula itself is obtained by analysis of a geometrical construction like that illustrated in Fig. 4 for 7 slits, each of which contributes a wave of amplitude a, with a phase difference of δ between adjacent slits.

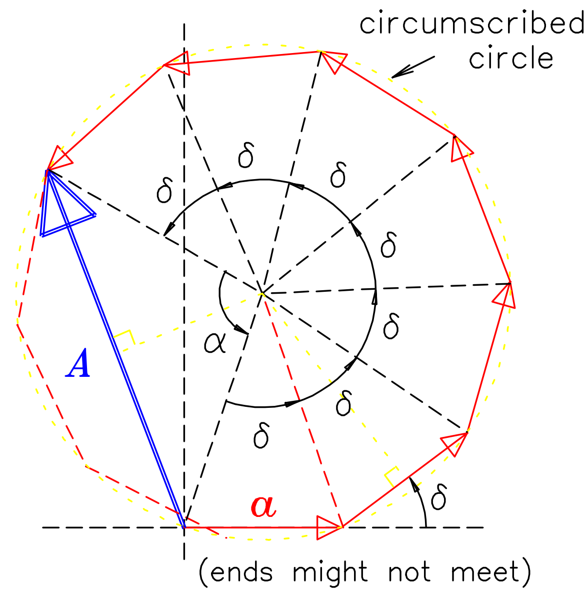

Figure 4 PHASOR DIAGRAM for calculating the intensity pattern produced by the interference of coherent light passing through 7 parallel, equally spaced slits.

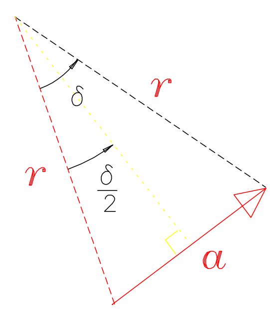

Figure 5 Blowup of one of the isosceles triangles formed by a single phasor and two radii from the center of the circumscribed circle to the tip and tail of the phasor.

After adding all 7 equal-length phasors in Fig. 4 "tip-to-tail", we can draw a vector from the starting point to the tip of the final phasor. This vector has a length A (the net amplitude) and makes a chord of the circumscribed circle, intercepting an angle

$$\alpha = 2\pi - N\,\delta\,, \tag{3}$$

where in this case $N = 7$. The radius r of the circumscribed circle is given by

$$\frac{a}{2} = r \sin\left(\frac{\delta}{2}\right) , \tag{4}$$

as can be seen from the blowup in Fig. 5; this can be combined with the analogous

$$\frac{A}{2} = r \sin\left(\frac{\alpha}{2}\right) \tag{5}$$

to give the net amplitude

$$A = a \left[\frac{\sin\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\delta}{2}\right)}\right] . \tag{6}$$

From Eq. (3) we know that $\alpha/2 = \pi - N\,\delta/2$, and in general $\sin(\pi - \theta) = \sin\theta$, so

$$\boxed{A = a \left[\frac{\sin\left(N\frac{\delta}{2}\right)}{\sin\left(\frac{\delta}{2}\right)}\right]} \tag{7}$$

where

$$\boxed{\delta = 2\pi\left(\frac{d}{\lambda}\right) \sin\vartheta} \tag{8}$$

Although the drawing shows $N = 7$ phasors, this result is valid for an arbitrary number N of equally spaced and evenly illuminated slits.

Figure 6 shows an example using 6 identical slits with a spacing $d = 100\lambda$. The angular width of the interference pattern from such widely spaced slits is quite narrow, only 10 mrad (10^{-2} radians) between principal maxima where all 6 rays are in phase. In between the principal maxima there are 5 minima and 4 secondary maxima; this can be generalized:

> The interference pattern for N equally spaced slits exhibits $(N-1)$ *minima* and $(N-2)$ *secondary maxima* between each pair of principal maxima.

We can understand this analytically from examining Eq. (7) with calculus: the extrema (maxima and minima) of A occur at the physical angles ϑ where $dA/d\vartheta = 0$. That is, where

$$\frac{\pi a d}{\lambda \sin\left(\frac{\delta}{2}\right)} \left[N \tan\left(\frac{\delta}{2}\right) - \tan\left(N\frac{\delta}{2}\right)\right] \cos\vartheta = 0 .$$

which is satisfied "trivially" at the CENTRAL MAXIMUM ($\vartheta = 0$) and otherwise where

$$\boxed{N \tan\left(\frac{\delta}{2}\right) = \tan\left(N\frac{\delta}{2}\right)} \tag{9}$$

The "almost trivial" solution is when $\delta/2$ is a multiple of π (or δ is a multiple of 2π), making both tangents zero. This corresponds to the PRINCIPAL MAXIMA where, as expected from Eq. (8),

$$\boxed{d \sin\vartheta_m = m\lambda} \tag{10}$$

What about the minima? From Fig. 6 we can see that the first minimum occurs when $N\delta = 2\pi$, that is, when the phasor diagram closes on itself. For large N this means that $\delta/2 = \pi/N$ is a rather small angle and we can use the small-angle approximation $\tan(\delta/2) \approx \delta/2 = \pi/N$ giving $N\tan(\delta/2) = \pi = N\delta/2$ to satisfy Eq. (9). Combined with Eq. (8) this predicts that the first minimum occurs at a laboratory angle $\vartheta(1^{\text{st}}\ \text{min.})$ given by

$$\boxed{d \sin\vartheta(1^{\text{st}}\ \text{min.}) = \frac{\lambda}{N}} \tag{11}$$

For large N and/or $d \gg \lambda$, this means

> The interference pattern for N equally spaced slits has its *first minimum* at an angle $\vartheta(1^{\text{st}}\ \text{min.})$ inversely proportional to N.

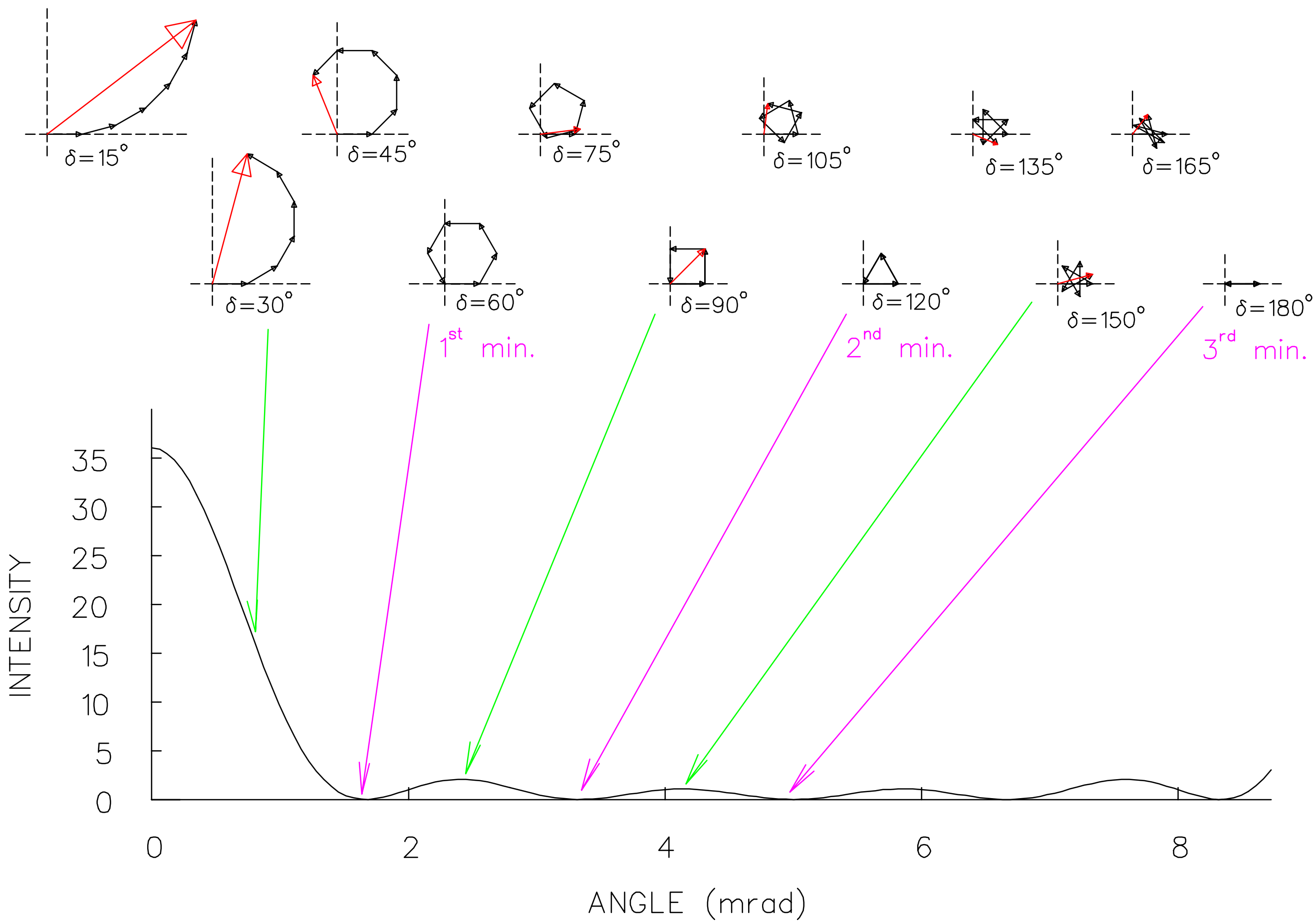

Figure 6 The intensity pattern produced by the interference of coherent light passing through six parallel slits 100 wavelengths apart. PHASOR DIAGRAMS are shown for selected angles. Note that, while the *phase* angle difference δ between rays from adjacent slits is a monotonically increasing function of the angle ϑ (plotted horizontally) that the rays make with the "forward" direction, the latter is a real geometrical angle in space while the former is a pure abstraction in "phase space". The exact relationship is $\delta/2\pi = (d/\lambda)\,\sin\vartheta \approx (d/\lambda)\,\vartheta$ for very small ϑ. Note the symmetry about the $3^{\rm rd}$ minimum at $\vartheta \approx 5$ mrad. At $\vartheta \approx 10$ mrad the intensity is back up to the same value it had in the central maximum at $\vartheta = 0$; this is called the first PRINCIPAL MAXIMUM. Then the whole pattern repeats. . . .